Announcements

For 11.29

1. HW12 is due on Thursday
    - HW12 is a practice final exam + some formal proofs
    - It is posted on Bb
2. Grades on Bb should be up to date
    - **Double check them!!**
3. **Final Exam**:
    - Take-home portion given out on Thurs 12.01
    - Take-home portion due at in-class final exam on Mon 12.12 (9-11:30am)

Formal Proofs and Logical Theory

Quantifiers and Completeness

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11.29.11

Outline

1. Quantificational Proofs
2. A Bit of Logical Theory

$\forall$ **Elim**

Universal Elimination

Universal Elimination (Official Version, Informal Step)

If you have $\forall x\, S(x)$, you may infer $S(c)$, provided that ‘c’ refers to an object in the domain of discourse

$\forall$ Elim

$\forall x\, S(x)$

$\vdots$

$\triangleright$ $S(c)$

EXAMPLE:

1	$\forall x\,(\mathrm{Tet}(x) \land \mathrm{Small}(x))$	
2	$\mathrm{Tet}(c) \land \mathrm{Small}(c)$	$\forall$ **Elim**: 1

∀ **Intro**

Versions 1 & 2

∀ **Intro** Version 1 (Formal Method of Proof)

$\boxed{c}$

⋮

$P(c)$

▷ $\forall x\, P(x)$

c cannot occur outside subproof where it is introduced: c must be arbitrary

- Boxed ‘c’ : *let c be arbitrary*

∀ **Intro** Version 2 (Formal Method of Proof)

$\boxed{c}\ A(c)$

⋮

$B(c)$

▷ $\forall x\, (A(x) \to B(x))$

c cannot occur outside subproof where it is introduced: c must be arbitrary

- Boxed ‘c’ : *let c be arbitrary*

An Example

∀ **Elim** and ∀ **Intro** at Work

Exercise 13.5 in Fitch.

Existential Intro

From Informal to Formal

Existential Introduction (Official Version, Informal Step)

If you have S(c), you may infer ∃x S(x), provided that ‘c’ refers to an object in the domain of discourse

∃ **Intro**

$S(c)$

⋮

▷ $\exists x\, S(x)$

EXAMPLE:

1 | $Tet(c)$

2 | $\exists x\, Tet(x)$ ∃ **Intro**: 1

Existential Elim

From Informal to Formal

Method of Existential Elimination

1. Given ∃x S(x), you may select a dummy name, say c, and assume S(c); whatever you can show from S(c) follows from the existential claim
2. c must be a new name, i.e. one not already used in your proof

∃ **Elim**

$\exists x\, S(x)$

⋮

$\boxed{c}\ S(c)$

⋮

Q

▷ Q

c must not occur outside of the subproof where it is introduced

That is, c must be arbitrary

The boxed c is read: *let c be an arbitrary individual such that. . .*

∃ Rules

An Example

In Fitch:

$\exists x\,(\mathsf{Small}(x) \wedge \mathsf{Cube}(x))$

$\exists x\,\mathsf{Small}(x) \wedge \exists x\,\mathsf{Cube}(x)$

Preventing Fallacies

In $\mathcal{F}$

- We've introduced constraints about the names we are allowed to use with ∀ **Intro** & ∃ **Elim**
- It is worth reminding ourselves of the bad things that would happen without these constraints
- To do this, let's go through a pseudo-proof see why it doesn't work either
- Finally, we'll construct a counterexample to the argument in Tarski's World to confirm that this is a good thing

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A Pseudo Proof

The Constraints on ∀ **Intro** and ∃ **Elim** at Work

The pseudo-proof (Fitch), then the counterexample (TW):

$\forall x\,\exists y\,\mathsf{Adjoins}(x, y)$

$\exists y\,\forall x\,\mathsf{Adjoins}(x, y)$

Working in Harmony

How to Mix things Up

The Moral

When using ∀ **Intro** and ∃ **Elim** together make sure that you are obeying the constraints on constants. Otherwise, you will end up giving 'proofs' for invalid arguments. If that were possible, the whole project of writing proofs and giving reasons would be nonsense.

A Real Proof

The Constraints on ∀ **Intro** and ∃ **Elim** at Work

$$\begin{array}{|l} \exists y \forall x\, \mathsf{Adjoins}(x, y) \\ \hline \forall x \exists y\, \mathsf{Adjoins}(x, y) \end{array}$$

How are ∀ and ∃ Related?

Three Ways

How ∀ and ∃ Relate

1. $\exists y \forall x\, \mathsf{S}(x, y)$ entails $\forall x \exists y\, \mathsf{S}(x, y)$
    - But $\forall x \exists y\, \mathsf{S}(x, y)$ does not entail $\exists y \forall x\, \mathsf{S}(x, y)$
2. $\forall x\, \mathsf{S}(x)$ entails $\exists x\, \mathsf{S}(x)$
    - But $\exists x\, \mathsf{S}(x)$ does not entail $\forall x\, \mathsf{S}(x)$
3. $\exists x \neg \mathsf{S}(x)$ is equivalent to $\neg \forall x\, \mathsf{S}(x)$
    - *Something is a non-cube* is the same as *Not everything is a cube*
    - Textbook shows: $\neg \forall x\, \mathsf{S}(x)$ entails $\exists x \neg \mathsf{S}(x)$
    - We'll show: $\exists x \neg \mathsf{S}(x)$ entails $\neg \forall x\, \mathsf{S}(x)$

Soundness & Completeness

Some Theoretical Questions

- Discussing the topic of being able to 'prove' invalid arguments raises two questions:
    1. If C is provable in $\mathcal{F}$ from $\mathsf{P}_1, \ldots, \mathsf{P}_\mathsf{n}$, is C an FO-Consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_\mathsf{n}$?
    2. If C an FO-Consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_\mathsf{n}$, is C is provable in $\mathcal{F}$ from $\mathsf{P}_1, \ldots, \mathsf{P}_\mathsf{n}$?
- If the answer to the 1st question is *yes* (*no*) we say that $\mathcal{F}$ is *sound* (*unsound*)
- If the answer to the 2nd question is *yes* (*no*) we say that $\mathcal{F}$ is *complete* (*incomplete*)

Interesting Fact

$\mathcal{F}$ is both sound and complete.

(This is proven in Chapters 18 & 19 of *LPL*)

Incompleteness of Number Theory

Gödel

- Kurt Gödel was the first to prove that FOL is sound and complete
- He later asked:
    - Suppose we make the following changes to FOL:
        1. Make the domain of discourse be the natural numbers $(0, 1, 2, 3, \ldots)$
        2. Add to $\mathcal{F}$ rules that characterize the operations of arithmetic $(<, +, \times)$
        3. Add the constant $\mathsf{0}$ and the function symbol S (*is the successor of*) and also add rules to $\mathcal{F}$ which characterize these

      Will the resulting system still be sound and complete?
- In 1931 he published an astonishing and brilliant proof which showed that although this system was sound, it was not complete!

Incompleteness

More on Gödel's Result

- This was a surprising result, since it would seem that Gödel's expanded version of FOL is exactly the version of FOL that we would use to formalize mathematical proofs about the nature of numbers
- But Gödel's result shows that not every valid argument can be formalized in this way!
- What does this mean about mathematical proof?
- Philosophers and mathematicians have spent a lot of time trying to get clear on this question
- If you are interested in this topic, I recommend reading Torkel Franzen's book *Gödel's Theorem*
    - And taking Phil 3310