

Announcements

For 11.22

- ① HW11 is due **now**
- ② Only 1 HW left!
 - HW12 is a practice final exam + some formal proofs
 - It will be posted on Bb today
 - It is not due until **Thursday 12.01**
- ③ Grades on Bb should be up to date
 - **Double check them!!**
- ④ **Final Exam:**
 - Take-home portion given out on **Thurs 12.01**
 - Take-home portion due at in-class final exam on **Mon 12.12 (9-11:30am)**

Formal Proofs for Quantifiers

 \exists Intro and \exists Elim

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11.22.11

Outline

- ① Review
- ② Formal Rules for \exists
- ③ Mixing Proof Methods

\forall Elim

Universal Elimination

Universal Elimination (Official Version, Informal Step)

If you have $\forall x S(x)$, you may infer $S(c)$, provided that 'c' refers to an object in the domain of discourse

\forall Elim

$\forall x S(x)$	
⋮	
$\triangleright S(c)$	

EXAMPLE:

1	$\forall x (Tet(x) \wedge Small(x))$	
2	$Tet(c) \wedge Small(c)$	\forall Elim: 1

\forall Intro

Versions 1 & 2

\forall Intro Version 1 (Formal Method of Proof)

<div style="border: 1px solid black; padding: 2px; display: inline-block; margin-bottom: 5px;">c</div> \vdots P(c)	<p style="color: red;">c cannot occur outside subproof where it is introduced: c must be arbitrary</p> <ul style="list-style-type: none"> • Boxed 'c': let c be arbitrary
$\triangleright \forall x P(x)$	

\forall Intro Version 2 (Formal Method of Proof)

<div style="border: 1px solid black; padding: 2px; display: inline-block; margin-bottom: 5px;">c</div> A(c) \vdots B(c)	<p style="color: red;">c cannot occur outside subproof where it is introduced: c must be arbitrary</p> <ul style="list-style-type: none"> • Boxed 'c': let c be arbitrary
$\triangleright \forall x (A(x) \rightarrow B(x))$	

\forall Intro

An Example

1	$\forall x [\text{Tet}(x) \rightarrow \neg \text{Cube}(x)]$	
2	$\forall x [\neg \text{Cube}(x) \rightarrow \text{Small}(x)]$	
3	<div style="border: 1px solid black; padding: 2px; display: inline-block; margin-bottom: 5px;">c</div> Tet(c)	
4	Tet(c) \rightarrow \neg Cube(c)	\forall Elim: 1
5	\neg Cube(c)	\rightarrow Elim: 3,4
6	\neg Cube(c) \rightarrow Small(c)	\forall Elim: 2
7	Small(c)	\rightarrow Elim: 5,6
8	$\forall x [\text{Tet}(x) \rightarrow \text{Small}(x)]$	\forall Intro: 3-7

Hint: working backwards is very helpful w/ \forall Intro

Overview

This Section

- For existential quantification, we've learned one informal inference step & one informal proof method:
 - 1 Existential Introduction
 - 2 The Method of Existential Elimination
- Today, we'll learn the formal counterparts of these informal principles
- But first, let's review the informal principles

Informal Steps & Methods

With the Existential Quantifier

Existential Introduction (Official Version)

From $S(n)$ you can infer $\exists x S(x)$, provided 'n' names an individual in the domain of discourse

Method of Existential Elimination

- 1 Given $\exists x S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c, and then **assume** $S(c)$
- 2 However, c must be a **new name**, i.e. one not already in use in the context of your proof

An Example

Using Existential Introduction and Elimination

$$\forall x (\text{Cube}(x) \rightarrow \text{Small}(x))$$

$$\exists x \text{Cube}(x)$$

$$\exists x \text{Small}(x)$$

Proof: We'll start by using existential elimination on premise two: let c be some cube. From premise 1 it follows by universal elimination that $\text{Cube}(c) \rightarrow \text{Small}(c)$. By modus ponens, c must be small. But then it follows that something is small (by existential introduction).

- Note that I applied existential elimination **before** universal elimination

Existential Intro

From Informal to Formal

Existential Introduction (Official Version, Informal Step)

If you have $S(c)$, you may infer $\exists x S(x)$, provided that ' c ' refers to an object in the domain of discourse

 \exists Intro
$$\begin{array}{|l} S(c) \\ \vdots \\ \hline \triangleright \exists x S(x) \end{array}$$

EXAMPLE:

$$\begin{array}{|l} 1 \quad \text{Tet}(c) \\ 2 \quad \hline \exists x \text{Tet}(x) \end{array} \quad \exists \text{Intro: } 1$$

Existential Elim

From Informal to Formal

Method of Existential Elimination

- Given $\exists x S(x)$, you may select a dummy name, say c , and assume $S(c)$; whatever you can show from $S(c)$ follows from the existential claim
- c must be a new name, i.e. one not already used in your proof

 \exists Elim
$$\exists x S(x)$$

$$\vdots$$

$$\boxed{c} S(c)$$

$$\vdots$$

$$Q$$

$$\triangleright Q$$

c must not occur outside of the subproof where it is introduced

That is, c must be arbitrary

The boxed c is read: *let c be an arbitrary individual such that...*

 \exists Rules

An Example

$$\begin{array}{|l} 1 \quad \forall x (\text{Cube}(x) \rightarrow \text{Small}(x)) \\ 2 \quad \exists x \text{Cube}(x) \\ 3 \quad \hline \boxed{c} \text{Cube}(c) \\ 4 \quad \text{Cube}(c) \rightarrow \text{Small}(c) \quad \forall \text{Elim: } 1 \\ 5 \quad \text{Small}(c) \quad \rightarrow \text{Elim: } 3, 4 \\ 6 \quad \exists x \text{Small}(x) \quad \exists \text{Intro: } 5 \\ 7 \quad \exists x \text{Small}(x) \quad \exists \text{Elim: } 2, 3-6 \end{array}$$

\exists Rules

Another Example

Let's do exercise 13.12. In this chapter we are free to use **Taut Con** to justify proof steps involving only propositional connectives.

$$\begin{array}{l}
 \mathbf{13.12} \quad \left| \begin{array}{l} \forall x (\text{Cube}(x) \vee \text{Tet}(x)) \\ \exists x \neg \text{Cube}(x) \\ \hline \exists x \text{Tet}(x) \end{array} \right.
 \end{array}$$

\exists Rules

More Practice

One more exercise in Fitch:

- Exercise 13.14

\exists Rules

In-Class Exercise

Construct a formal proof for the following argument.

$$\begin{array}{l}
 1 \quad \left| \forall x (\text{Tet}(x) \rightarrow \text{Medium}(x)) \right. \\
 2 \quad \left| \exists y \text{Tet}(y) \right. \\
 \hline
 3 \quad \left| \exists x (\text{Tet}(x) \wedge \text{Medium}(x)) \right.
 \end{array}$$

Mixing Quantifiers

A Pseudo-Proof

$$\begin{array}{l}
 1 \quad \left| \forall x \exists y \text{Loves}(x, y) \right. \\
 2 \quad \left| \exists y \forall x \text{Loves}(x, y) \right.
 \end{array}$$

Pseudo-Proof: The premise says that everyone likes someone or other. Let b be any boy, then there's a girl he loves. Call her g . **Since b was arbitrary**, we may conclude by Univ. Intro. that $\forall x \text{Loves}(x, g)$. By Exist. Intro. our conclusion follows.

- The crucial misstep: claim that b was arbitrary
- g is a girl that b likes, so b ceases to be arbitrary
 - Our proof then contains specific information about b : that he likes g , which is **not** true of everyone

Mixing Quantifiers

A Pseudo-Proof

1	$\forall x \exists y \text{ Loves}(x, y)$	
2	b	
3	$\exists y \text{ Loves}(b, y)$	\forall Elim : 1
4	g $\text{Loves}(b, g)$	
5	$\text{Loves}(b, g)$	Reit: 4
6	$\text{Loves}(b, g)$	\exists Elim : 3,4-5 × × ×
7	$\forall x \text{ Loves}(x, g)$	\forall Intro : 6 × × ×
8	$\exists y \forall x \text{ Loves}(x, y)$	\exists Intro : 7

Lines 6 & 7: g occurs outside of subproof of introduction

Mixing Quantifiers

The Moral

Summary

- ① **Universal Introduction:** the arbitrary constant selected must not occur anywhere outside the subproof in which it is introduced.
- ② **Existential Elimination:** the arbitrary constant selected must not occur anywhere outside the subproof in which it is introduced
 - Above, g occurred outside of the subproof in which it was introduced!
- ③ It was nuanced to see exactly how to manage arbitrary constants in informal proofs
- ④ But in formal proofs, subproofs give us the resources to state these conditions precisely