Announcements
For 11.22

1. HW11 is due now
2. Only 1 HW left!
   - HW12 is a practice final exam + some formal proofs
   - It will be posted on Bb today
   - It is not due until Thursday 12.01
3. Grades on Bb should be up to date
   - Double check them!!
4. Final Exam:
   - Take-home portion given out on Thurs 12.01
   - Take-home portion due at in-class final exam on Mon 12.12 (9-11:30am)

Formal Proofs for Quantifiers
\( \exists \) Intro and \( \exists \) Elim

William Starr
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Outline

1. Review
2. Formal Rules for \( \exists \)
3. Mixing Proof Methods

Universal Elimination (Official Version, Informal Step)
If you have \( \forall x S(x) \), you may infer \( S(c) \), provided that ‘c’ refers to an object in the domain of discourse

\( \forall \text{ Elim} \)
\[
\begin{align*}
&\forall x S(x) \\
\vdash & S(c)
\end{align*}
\]

Example:
\[
\begin{align*}
1 & \forall x (Tet(x) \land Small(x)) \\
2 & Tet(c) \land Small(c) \quad \forall \text{ Elim: 1}
\end{align*}
\]
\[\forall\text{ Intro Version 1 (Formal Method of Proof)}\]

- **C**
  - \(c\) cannot occur outside subproof where it is introduced: \(c\) must be arbitrary
  - Boxed ‘\(c\)’: *let \(c\) be arbitrary*

\[\forall P(c) \implies \forall x P(x)\]

\[\forall\text{ Intro Version 2 (Formal Method of Proof)}\]

- **C**
  - \(A(c)\)
  - \(B(c)\)
  - Boxed ‘\(c\)’: *let \(c\) be arbitrary*

\[\forall x (A(x) \implies B(x))\]

**Overview**

This Section

- For existential quantification, we’ve learned one informal inference step & one informal proof method:
  1. Existential Introduction
  2. The Method of Existential Elimination
- Today, we’ll learn the formal counterparts of these informal principles
- But first, let’s review the informal principles

**Informal Steps & Methods**

With the Existential Quantifier

- **Existential Introduction (Official Version)**
  - From \(S(n)\) you can infer \(\exists x S(x)\), provided ‘\(n\)’ names an individual in the domain of discourse

- **Method of Existential Elimination**
  - Given \(\exists x S(x)\), you may give a dummy name to (one of) the object(s) satisfying \(S(x)\), say \(c\), and then assume \(S(c)\)
  - However, \(c\) must be a **new name**, i.e. one not already in use in the context of your proof

**An Example**

1. \(\forall x [\text{Tet}(x) \implies \neg \text{Cube}(x)]\)
2. \(\forall x [\neg \text{Cube}(x) \implies \text{Small}(x)]\)
3. \(\forall x \text{Tet}(c)\)
4. \(\text{Tet}(c) \implies \neg \text{Cube}(c)\) \(\quad \forall\text{ Elim: 1}\)
5. \(\neg \text{Cube}(c)\) \(\quad \rightarrow\text{ Elim: 3, 4}\)
6. \(\neg \text{Cube}(c) \implies \text{Small}(c)\) \(\quad \forall\text{ Elim: 2}\)
7. \(\text{Small}(c)\) \(\quad \rightarrow\text{ Elim: 5, 6}\)
8. \(\forall x [\text{Tet}(x) \implies \text{Small}(x)]\) \(\quad \forall\text{ Intro: 3-7}\)

**Hint**: working backwards is very helpful w/ \(\forall\text{ Intro} \)
Review Formal Rules for ∃ Mixing Proof Methods

An Example
Using Existential Introduction and Elimination

∀x (Cube(x) → Small(x))
∃x Cube(x)
∴ ∃x Small(x)

Proof: We’ll start by using existential elimination on premise two: let c be some cube. From premise 1 it follows by universal elimination that Cube(c) → Small(c) By modus ponens, c must be small. But then it follows that something is small (by existential introduction).

• Note that I applied existential elimination before universal elimination

Existential Intro
From Informal to Formal

Existential Introduction (Official Version, Informal Step)
If you have S(c), you may infer ∃x S(x), provided that ‘c’ refers to an object in the domain of discourse

∃ Intro

Example:

S(c)
∴ ∃x S(x)

Method of Existential Elimination

1. Given ∃x S(x), you may select a dummy name, say c, and assume S(c); whatever you can show from S(c) follows from the existential claim
2. c must be a new name, i.e. one not already used in your proof

∃ Elim

| ∃x S(x) |
| : |
| c must not occur outside of the subproof where it is introduced |
| [c] S(c) |
| : |
| That is, c must be arbitrary |
| Q |
| : |
| The boxed c is read: let c be an arbitrary individual such that . . . |
| □ |
| Q |

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Existential Elim
From Informal to Formal

∃ Rules
An Example

| 1 | ∀x (Cube(x) → Small(x)) |
| 2 | ∃x Cube(x) |
| 3 | [c] Cube(c) |
| 4 | Cube(c) → Small(c) | ∀ Elim: 1 |
| 5 | Small(c) | → Elim: 3, 4 |
| 6 | ∃x Small(x) | ∃ Intro: 5 |
| 7 | ∃x Small(x) | ∃ Elim: 2, 3-6 |

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Let’s do exercise 13.12. In this chapter we are free to use Taut Con to justify proof steps involving only propositional connectives.

13.12

\[
\begin{align*}
\forall x (\text{Cube}(x) \lor \text{Tet}(x)) \\
\exists x \lnot \text{Cube}(x) \\
\exists x \text{Tet}(x)
\end{align*}
\]

Construct a formal proof for the following argument.

1. \( \forall x (\text{Tet}(x) \rightarrow \text{Medium}(x)) \)
2. \( \exists y \text{Tet}(y) \)
3. \( \exists x (\text{Tet}(x) \land \text{Medium}(x)) \)

One more exercise in Fitch:
- Exercise 13.14

Pseudo-Proof: The premise says that everyone likes someone or other. Let \( b \) be any boy, then there’s a girl he loves. Call her \( g \). Since \( b \) was arbitrary, we may conclude by Univ. Intro. that \( \forall x \text{Loves}(x, g) \). By Exist. Intro. our conclusion follows.
- The crucial misstep: claim that \( b \) was arbitrary
- \( g \) is a girl that \( b \) likes, so \( b \) ceases to be arbitrary
- Our proof then contains specific information about \( b \): that he likes \( g \), which is not true of everyone
Mixing Quantifiers

A Pseudo-Proof

1. \(\forall x \exists y \text{ Loves}(x, y)\)
2. \(\exists y \text{ Loves}(b, y)\) \(\forall\ \text{Elim}:\ 1\)
3. \(\exists y \text{ Loves}(b, y)\) \(\forall\ \text{Elim}:\ 1\)
4. \(\forall x \text{ Loves}(b, g)\)
5. \(\text{Loves}(b, g)\) \(\text{Reit}:\ 4\)
6. \(\text{Loves}(b, g)\) \(\exists\ \text{Elim}:\ 3, 4-5 \times \times \times\)
7. \(\forall x \text{ Loves}(x, g)\) \(\forall\ \text{Intro}:\ 6 \times \times \times\)
8. \(\exists y \forall x \text{ Loves}(x, y)\) \(\exists\ \text{Intro}:\ 7\)

Lines 6 & 7: \(g\) occurs outside of subproof of introduction

Summary

1. **Universal Introduction**: the arbitrary constant selected must not occur anywhere outside the subproof in which it is introduced.
2. **Existential Elimination**: the arbitrary constant selected must not occur anywhere outside the subproof in which it is introduced
   - Above, \(g\) occurred outside of the subproof in which it was introduced!
3. It was nuanced to see exactly how to manage arbitrary constants in informal proofs
4. But in formal proofs, subproofs give us the resources to state these conditions precisely