Formal Proofs for Quantifiers

∀ Intro and **∀ Elim**

William Starr

11.17.11

Announcements

For 11.17

1. Check your grades on Bb!!
2. Follow instructions from last announcement if you have unexpected 0s

Outline

Summary of Informal Steps & Methods

Overview

- For universal quantification, we've learned one informal inference step & two informal proof methods:
  - Universal Elimination
  - Universal Introduction
  - General Conditional Proof

Summary of Informal Steps & Methods

Universal Elimination (Official Version)

If you have $\forall x\, \mathsf{S}(x)$, you may infer $\mathsf{S}(c)$, provided that 'c' refers to an object in the domain of discourse

Universal Introduction

To prove $\forall x\, \mathsf{S}(x)$:

1. Introduce a new name c to stand for a completely arbitrary member of the domain of discourse
2. Prove $\mathsf{S}(c)$
3. Conclude $\forall x\, \mathsf{S}(x)$

An Example

Using Universal Elimination & Introduction

$$\begin{array}{l|l} 1 & \forall x\, \mathsf{Tet}(x) \\ \hline 2 & \forall x\, (\mathsf{Tet}(x) \lor \mathsf{Small}(x)) \end{array}$$

Proof: We will use universal introduction to prove the conclusion. Let c name an arbitrary block. From 1 $\mathsf{Tet}(c)$ follows by universal elimination. But if *that* is true, so is $\mathsf{Tet}(c) \lor \mathsf{Small}(c)$ (by disjunction introduction). Since c was arbitrary, it follows that $\forall x\, (\mathsf{Tet}(x) \lor \mathsf{Small}(x))$.

Summary of Informal Steps & Methods

General Conditional Proof

General Conditional Proof

To prove $\forall x\, (\mathsf{A}(x) \to \mathsf{B}(x))$:

1. Introduce a new arbitrary name c, and assume $\mathsf{A}(c)$
2. Prove $\mathsf{B}(c)$
3. Conclude $\forall x\, (\mathsf{A}(x) \to \mathsf{B}(x))$

- This is equivalent to using universal introduction *and* conditional proof together

An Example

General Conditional Proof

$$\begin{array}{l|l} 1 & \mathsf{Medium}(a) \\ \hline 2 & \forall y\, (\mathsf{Smaller}(a, y) \to \mathsf{Large}(y)) \end{array}$$

Proof: We will use general conditional proof to show our conclusion. Let c stand for an arbitrary block and assume for the sake of argument that $\mathsf{Smaller}(a, c)$. We will try to show that $\mathsf{Large}(c)$. From 1 we know that a is medium. But our assumption tells us that a is smaller than c, so c must be large. This is exactly what we were out to show to finish our general conditional proof, so we are done.

Formal Proofs

Getting Back Into the Swing of It

- Remember formal proofs?
  - Contains only wffs of FOL
  - Every non-premise or assumption line must cite a formal rule as its justification
- Example:

1	$\mathsf{Tet}(a) \to \mathsf{Smaller}(a, c)$	
2	$\quad \mathsf{Tet}(a)$	
3	$\quad \mathsf{Smaller}(a, c)$	$\to$ **Elim**: 2, 1
4	$\quad \mathsf{Smaller}(a, c) \lor \mathsf{LeftOf}(a, b)$	$\lor$ **Intro**: 3
5	$\mathsf{Tet}(a) \to (\mathsf{Smaller}(a, c) \lor \mathsf{LeftOf}(a, b))$	$\to$ **Intro**: 2-4

Formal Proofs

Review

- Remember: every logical symbol gets two rules:
  1. = **Elim**, = **Intro**
  2. ¬ **Elim**, ¬ **Intro**
  3. ⊥ **Elim**, ⊥ **Intro**
  4. ∧ **Elim**, ∧ **Intro**
  5. ∨ **Elim**, ∨ **Intro**
  6. → **Elim**, → **Intro**
  7. ↔ **Elim**, ↔ **Intro**
- Recall that each of these rules was the formal analog of an informal inference step or proof method
- The rules are justified by the meaning of each symbol
- Now, we'll learn the formal rules for ∀: ∀ **Intro**, ∀ **Elim**

∀ Elim

Universal Elimination

Universal Elimination (Official Version, Informal Step)

If you have $\forall x\, \mathsf{S}(x)$, you may infer $\mathsf{S}(c)$, provided that 'c' refers to an object in the domain of discourse

∀ Elim

	$\forall x\, \mathsf{S}(x)$
	$\vdots$
▷	$\mathsf{S}(c)$

EXAMPLE:

1	$\forall x\, (\mathsf{Tet}(x) \land \mathsf{Small}(x))$	
2	$\mathsf{Tet}(c) \land \mathsf{Small}(c)$	∀ **Elim**: 1

∀ Intro

A Preliminary Note

- Two methods of informally proving universal claims:
  1. Universal Introduction
  2. General Conditional Proof
- We will consider two versions of ∀ **Intro** corresponding to each of these methods

∀ **Intro**
Version 1

∀ Intro Version 1 (Formal Method of Proof)

$\boxed{c}$	
⋮	
P(c)	
▷ ∀x P(x)	

c must not occur outside of the subproof where it is introduced

That is, c must be arbitrary

- Read the boxed 'c' as *let* c *be an arbitary member of the domain of discourse*

∀ **Intro**, Version 1
An Example

1	∀x (Tet(x) ∧ Small(x))	
2	$\boxed{c}$	
3	Tet(c) ∧ Small(c)	∀ **Elim**: 1
4	Tet(c)	∧ **Elim**: 3
5	∀x Tet(x)	∀ **Intro**: 2-4

∀ **Intro**, Version 1
Another Example

1	∀x Tet(x)
2	∀x (Tet(x) ∨ Small(x))

Proof: We will use universal introduction to prove the conclusion. Let c name an arbitrary block. From 1 Tet(c) follows by universal elimination. But if *that* is true, so is Tet(c) ∨ Small(c) (by disjunction introduction). Since c was arbitrary, it follows that ∀x (Tet(x) ∨ Small(x)).

1	∀x Tet(x)	
2	$\boxed{c}$	
3	Tet(c)	∀ **Elim**: 1
4	Tet(c) ∨ Small(c)	∨ **Intro**: 3
5	∀x (Tet(x) ∨ Small(x))	∀ **Intro**: 2-4

∀ **Intro**, Version 1
Using Fitch

- Let's do **Exercise 13.2**
- But, we'll learn how to use Fitch to do it

13.2

∀x (Cube(x) ↔ Small(x))

∀x Cube(x)

∀x Small(x)

∀ Intro

In Class Exercise

Give a *formal proof* that the following argument is valid

$\forall y\, \mathsf{Small}(y)$

$\forall x\, \mathsf{Cube}(x)$

$\forall x\, (\mathsf{Cube}(x) \wedge \mathsf{Small}(x))$

It may be helpful to first do an informal proof

Hint: Since you are trying to prove a universal claim, use the method of ∀ **Intro**

∀ Intro

The Rule, Version 2

General Conditional Proof (Informal Method of Proof)

1. Introduce a new name c to stand for a completely arbitrary member of the domain of discourse and assume $\mathsf{A}(\mathsf{c})$
2. Prove $\mathsf{B}(\mathsf{c})$
3. Conclude $\forall x\, (\mathsf{A}(x) \rightarrow \mathsf{B}(x))$

∀ Intro Version 2 (Formal Method of Proof)

	$\boxed{\mathsf{c}}\ \mathsf{A}(\mathsf{c})$
	$\vdots$
	$\mathsf{B}(\mathsf{c})$
▷	$\forall x\, (\mathsf{A}(x) \rightarrow \mathsf{B}(x))$

c must not occur outside of the subproof where it is introduced

That is, c must be arbitrary

- Read the boxed 'c' as *let* c *be an arbitary member of the domain of discourse*

∀ Intro

General Conditional Proof

1	$\mathsf{Medium}(a)$	
2	$\forall y\, (\mathsf{Smaller}(a, y) \rightarrow \mathsf{Large}(y))$	

Proof: Let c be an arbitrary block & assume $\mathsf{Smaller}(a, c)$. Since our premise says that a is medium, our assumption requires that $\mathsf{Large}(c)$. Because c was arbitrary, the conclusion follows by General Conditional Proof.

1	$\mathsf{Medium}(a)$	
2	$\boxed{\mathsf{c}}\ \mathsf{Smaller}(a, c)$	
3	$\mathsf{Large}(c)$	**Ana Con**: 1,2
4	$\forall y\, (\mathsf{Smaller}(a, y) \rightarrow \mathsf{Large}(y))$	∀ **Intro**: 2-3

∀ Intro

An Example

1	$\forall x\, [\mathsf{Tet}(x) \rightarrow \neg \mathsf{Cube}(x)]$	
2	$\forall x\, [\neg \mathsf{Cube}(x) \rightarrow \mathsf{Small}(x)]$	
3	$\boxed{\mathsf{c}}\ \mathsf{Tet}(c)$	
4	$\mathsf{Tet}(c) \rightarrow \neg \mathsf{Cube}(c)$	∀ **Elim**: 1
5	$\neg \mathsf{Cube}(c)$	→ **Elim**: 3,4
6	$\neg \mathsf{Cube}(c) \rightarrow \mathsf{Small}(c)$	∀ **Elim**: 2
7	$\mathsf{Small}(c)$	→ **Elim**: 5,6
8	$\forall x\, [\mathsf{Tet}(x) \rightarrow \mathsf{Small}(x)]$	∀ **Intro**: 3-7

Hint: working backwards is very helpful w/∀ **Intro**

Another Example
In Fitch

Time permitting, in Fitch:

1. Exercise 13.7

Summary
For 11.17

Summary

1. We learned how to use ∀ **Elim** and ∀ **Intro**
   - ∀ **Intro** came in two versions corresponding to our two informal proof methods
   - Both versions used the notation of boxing constants where our informal proofs used ‘arbitrary names’
2. When using ∀ **Intro** it is very helpful to work backwards