Informal Proofs with Quantifiers III

Mixed Proofs

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Announcements

11.15

1. Grades for HW1-8 are on Bb
   - Check them!
2. Many people have been submitting electronic HW incorrectly
   - See recent announcement
   - If you have 0s or low scores on all electronic assignments: this applies to you!

Outline

1. Review
2. Caution When Mixing Quantifiers
3. Proofs With Mixed Quantifiers

Two Inference Steps

In Review

Existential Introduction (Official Version)

$$\begin{array}{l|l} & S(c) \\ \hline \triangleright & \exists x\, S(x) \end{array}$$

(When 'c' names an object in the domain of discourse)

Universal Elimination (Official Version)

$$\begin{array}{l|l} & \forall x\, S(x) \\ \hline \triangleright & S(c) \end{array}$$

(Where 'c' refers to an object in the domain of discourse)

Existential Elimination
In Review

The Method of Existential Elimination

1. Given $\exists x\, S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c, and then assume $S(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

- You are likely to need this method when you have an existential premise

Existential Elimination
An Example

Example Argument

1	$\forall x\, [Tet(x) \to Small(x)]$
2	$\exists x\, Tet(x)$
3	$\exists x\, Small(x)$

Proof:

- From 2 we know there is some block, call it d, such that $Tet(d)$ (Exist. Elim.)
- From 1 by Univ. Elim.:$Tet(d) \to Small(d)$

- So we have $Small(d)$ by modus ponens
- By Exist. Intro. it follows that: $\exists x\, Small(x)$ ✓

Universal Introduction
The Official Formulation

Universal Introduction

To prove $\forall x\, S(x)$:

1. Introduce a new name c to stand for a completely arbitrary member of the domain of discourse
2. Prove $S(c)$
3. Conclude $\forall x\, S(x)$

- You will need to use this method whenever you are trying to **prove** a universal claim
- You **do not** use the method 'on' universal **premises**

Universal Introduction
An Example

1	$\forall y\, SameSize(y, b)$
2	$\forall x\, [SameSize(x, b) \to LeftOf(x, a)]$
3	$\forall x\, \exists y\, LeftOf(x, y)$

Proof: Let c be an arbitrary block. (*Goal*: $\exists y\, LeftOf(c, y)$) From 1 we get $SameSize(c, b)$, by Univ. Elim. From 2 we get $SameSize(c, b) \to LeftOf(c, a)$. So $LeftOf(c, a)$ follows by modus ponens. By Exist. Intro. we get $\exists y\, LeftOf(c, y)$. But c was arbitrary, so $\forall x\, \exists y\, Smaller(x, y)$ follows by Univ. Intro.

General Conditional Proof
In Review

General Conditional Proof

To prove $\forall x\,(A(x) \rightarrow B(x))$:

1. Introduce a new name c to stand for a completely arbitrary member of the domain of discourse
2. Assume $A(c)$
3. Prove $B(c)$
4. Conclude $\forall x\,(A(x) \rightarrow B(x))$

- Use this method to **prove** universal conditionals like $\forall x\,(Cube(x) \rightarrow Small(x))$

Mixing Quantifiers
In a Proof

- We will often want to use both Exist. Elim. and Univ. Intro. or Gen. Cond. Pf.
- Two of the important facts we've recently learned:
  - Using existential elimination requires the careful use of arbitrary names
  - Using universal introduction requires the careful use of arbitrary names
- An equally important consequence of these facts:

Important Fact about Mixing Quantifier Proof Methods

Using existential elimination and universal introduction together requires doubly careful use of arbitrary names.

Mixing Quantifiers
A Real Proof

1 | $\exists y\, \forall x\, Loves(x, y)$

2 | $\forall x\, \exists y\, Loves(x, y)$

Proof: We will show that $\exists y\, Loves(a, y)$, holds for an arbitrary a. Given the premise, at least one person is loved by everyone. Assume d is one of these lucky people: $\forall x\, Loves(x, d)$, by Exist. Elim. Univ. Elim. gives us $Loves(a, d)$. By Exist. Intro. it follows that $\exists y\, Loves(a, y)$. Since a was arbitrary, it follows by Univ. Intro. that $\forall x\, \exists y\, Loves(x, y)$.

Mixing Quantifiers
The Opposite Inference is Invalid

1 | $\forall x\, \exists y\, Loves(x, y)$

2 | $\exists y\, \forall x\, Loves(x, y)$

- We know that this inference isn't valid
- Consider a world with two people:
  - Alice and Bob
  - Bob loves Alice
  - Alice loves Bob
  - But, Bob does not love himself
  - And Alice does not love herself
- The premise is true: everyone loves someone or other
- But the conclusion is false, no one is loved by everyone

Mixing Quantifiers
A Pseudo-Proof

1 | $\forall x\, \exists y\, \mathsf{Loves}(x, y)$

2 | $\exists y\, \forall x\, \mathsf{Loves}(x, y)$

Pseudo-Proof: Let b be an arbitrary boy. By premise 1, he loves some girl. Assume it's g. Since b was chosen arbitrarily, we may conclude by Univ. Intro. that $\forall x\, \mathsf{Loves}(x, g)$. The conclusion follows by exist. intro.!

- The crucial misstep: is b wasn't arbitrary!
- Why? Introducing g, a specific girl that b likes, makes b non-arbitrary!
  - Our proof then contains information particular to b: which girl he likes, namely g!

Mixing Quantifiers
The Moral

Summary

1. **Universal Introduction**: To prove $\forall x\, S(x)$, chose a new constant c and prove $S(c)$, making sure that $S(c)$ does not contain any names introduced by Exist. Elim. after the introduction of c.
   - Above, g was introduced by Exist. Elim. after b
2. Same for applications of General Conditional Proof

- If you do not follow this advice, you will be able to give 'proofs' of invalid arguments
- But, if there's proofs of invalid arguments, the whole idea of proof is bankrupt!

In-Class Exercise
Exercise 12.14

12.14 *This exercise contains a purported proof. If it is correct, say so. If it is incorrect, explain what goes wrong using the notions presented above.*

| $\exists x\, (x = x \rightarrow \neg\exists y\, x \neq y)$ (There is at most one object)

Purported proof: Toward a proof by contradiction, suppose $\neg\exists x\, (x = x \rightarrow \neg\exists y\, x \neq y)$. This is equivalent to $\forall x\, \neg(x = x \rightarrow \neg\exists y\, x \neq y)$, which is equivalent to $\forall x\, (x = x \wedge \exists y\, x \neq y)$. By Univ. Elim. we get $c = c \wedge \exists y\, c \neq y$. By Exist. Elim. we may assume $c \neq d$. But, since c was arbitrary, it follows that $\forall x\, x \neq d$. By Univ. Elim, we get $d \neq d$, which is a contradiction. Thus, the conclusion must be true.

Mixing Quantifiers in Proofs
Where We Are

- So far: an important lesson about how **not** to apply Univ. Intro. and Exist. Elim.
- We learned how to recognize this mistake
- But we also need to practice correctly mixing these two rules
- So let's do some more informal proofs that require mixing the two rules

Mixing Quantifiers
An Example

$\forall x \forall y\, [\text{Smaller}(x, y) \to \text{SameShape}(x, y)]$

$\forall x \exists y\, [\text{Adjoins}(x, y) \to \text{Smaller}(x, y)]$

$\forall x \exists y\, [\text{Adjoins}(x, y) \to \text{SameShape}(x, y)]$

Proof: From premise 2 by Univ. Elim. $\exists y\, [\text{Adjoins}(c, y) \to \text{Smaller}(c, y)]$. By Exist. Elim. we may then assume $\text{Adjoins}(c, d) \to \text{Smaller}(c, d)$. From premise 1 by Univ. Elim. $\text{Smaller}(c, d) \to \text{SameShape}(c, d)$. By the transitivity of $\to$, we have $\text{Adjoins}(c, d) \to \text{SameShape}(c, d)$. Exist. Intro. then gives us $\exists y\, [\text{Adjoins}(c, y) \to \text{SameShape}(c, y)]$. Since c was arbitrary, the conclusion follows by Univ. Intro.

Mixing Quantifiers
Another Example

$\exists y \forall x\, (\text{Smaller}(x, y) \lor \neg\text{Tet}(x))$

$\forall y \forall x\, (\text{Smaller}(y, x) \to \text{Cube}(y))$

$\forall z \forall y\, [(\text{LeftOf}(z, y) \land \neg\text{Tet}(z)) \to \text{Cube}(z)]$

$\forall x\, (\forall y\, \text{LeftOf}(x, y) \to \text{Cube}(x))$

Proof: We will use general conditional proof, but first we apply Exist. Elim. to premise 1 and assume $\forall x\, (\text{Smaller}(x, a) \lor \neg\text{Tet}(x))$. Now we take an arbitrary c and assume $\forall y\, \text{LeftOf}(c, y)$, with the goal of showing $\text{Cube}(c)$. This assumption gives us $\text{LeftOf}(c, a)$ by Univ. Elim. By Univ. Elim. we also have $\text{Smaller}(c, a) \lor \neg\text{Tet}(c)$. Consider the second case. Premise 3 gives us $(\text{LeftOf}(c, a) \land \neg\text{Tet}(c)) \to \text{Cube}(c)$. Then $\text{Cube}(c)$ follows by modus ponens. In the second case, premise 2 gives us $(\text{Smaller}(c, a) \to \text{Cube}(c))$ by Univ. Elim. So we have $\text{Cube}(c)$ again by modus ponens. Thus, either way, we have $\text{Cube}(c)$, and since c was arbitrary the conclusion follows by general conditional proof.