Informal Proofs with Quantifiers III

Mixed Proofs

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Announcements

11.15

1. Grades for HW1-8 are on Bb
   - Check them!
2. Many people have been submitting electronic HW incorrectly
   - See recent announcement
   - If you have 0s or low scores on all electronic assignments: this applies to you!

Outline

Two Inference Steps

In Review

Existential Introduction (Official Version)

$$\begin{array}{r|l} & \mathsf{S(c)} \\ \hline \triangleright & \exists x\, \mathsf{S(x)} \end{array}$$

(When ‘c’ names an object in the domain of discourse)

Universal Elimination (Official Version)

$$\begin{array}{r|l} & \forall x\, \mathsf{S(x)} \\ \hline \triangleright & \mathsf{S(c)} \end{array}$$

(Where ‘c’ refers to an object in the domain of discourse)

Existential Elimination

In Review

The Method of Existential Elimination

1. Given $\exists x\, \mathsf{S}(x)$, you may give a dummy name to (one of) the object(s) satisfying $\mathsf{S}(x)$, say c, and then assume $\mathsf{S}(\mathsf{c})$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

- You are likely to need this method when you have an existential premise

Existential Elimination

An Example

Example Argument

1 | $\forall x\, [\mathsf{Tet}(x) \rightarrow \mathsf{Small}(x)]$
2 | $\exists x\, \mathsf{Tet}(x)$
3 | $\exists x\, \mathsf{Small}(x)$

Proof:

- From 2 we know there is some block, call it d, such that $\mathsf{Tet}(\mathsf{d})$ (Exist. Elim.)
- From 1 by Univ. Elim.:$\mathsf{Tet}(\mathsf{d}) \rightarrow \mathsf{Small}(\mathsf{d})$

- So we have $\mathsf{Small}(\mathsf{d})$ by modus ponens
- By Exist. Intro. it follows that: $\exists x\, \mathsf{Small}(x)$ ✓

Universal Introduction

The Official Formulation

Universal Introduction

To prove $\forall x\, \mathsf{S}(x)$:

1. Introduce a new name c to stand for a completely arbitrary member of the domain of discourse
2. Prove $\mathsf{S}(\mathsf{c})$
3. Conclude $\forall x\, \mathsf{S}(x)$

- You will need to use this method whenever you are trying to **prove** a universal claim
- You **do not** use the method 'on' universal **premises**

Universal Introduction

An Example

1 | $\forall y\, \mathsf{SameSize}(y, b)$
2 | $\forall x\, [\mathsf{SameSize}(x, b) \rightarrow \mathsf{LeftOf}(x, a)]$
3 | $\forall x\, \exists y\, \mathsf{LeftOf}(x, y)$

Proof: Let c be an arbitrary block. (*Goal*: $\exists y\, \mathsf{LeftOf}(c, y)$) From 1 we get $\mathsf{SameSize}(c, b)$, by Univ. Elim. From 2 we get $\mathsf{SameSize}(c, b) \rightarrow \mathsf{LeftOf}(c, a)$. So $\mathsf{LeftOf}(c, a)$ follows by modus ponens. By Exist. Intro. we get $\exists y\, \mathsf{LeftOf}(c, y)$. But c was arbitrary, so $\forall x\, \exists y\, \mathsf{Smaller}(x, y)$ follows by Univ. Intro.

General Conditional Proof
In Review

General Conditional Proof

To prove $\forall x\,(A(x) \to B(x))$:

1. Introduce a new name c to stand for a completely arbitrary member of the domain of discourse
2. Assume $A(c)$
3. Prove $B(c)$
4. Conclude $\forall x\,(A(x) \to B(x))$

- Use this method to **prove** universal conditionals like $\forall x\,(\mathsf{Cube}(x) \to \mathsf{Small}(x))$

Mixing Quantifiers
In a Proof

- We will often want to use both Exist. Elim. and Univ. Intro. or Gen. Cond. Pf.
- Two of the important facts we've recently learned:
  - Using existential elimination requires the careful use of arbitrary names
  - Using universal introduction requires the careful use of arbitrary names
- An equally important consequence of these facts:

Important Fact about Mixing Quantifier Proof Methods

Using existential elimination and universal introduction together requires doubly careful use of arbitrary names.

Mixing Quantifiers
A Real Proof

$$\begin{array}{l|l} 1 & \exists y\,\forall x\,\mathsf{Loves}(x,y) \\ \hline 2 & \forall x\,\exists y\,\mathsf{Loves}(x,y) \end{array}$$

Proof: We will show that $\exists y\,\mathsf{Loves}(a,y)$, holds for an arbitrary a. Given the premise, at least one person is loved by everyone. Assume d is one of these lucky people: $\forall x\,\mathsf{Loves}(x,d)$, by Exist. Elim. Univ. Elim. gives us $\mathsf{Loves}(a,d)$. By Exist. Intro. it follows that $\exists y\,\mathsf{Loves}(a,y)$. Since a was arbitrary, it follows by Univ. Intro. that $\forall x\,\exists y\,\mathsf{Loves}(x,y)$.

Mixing Quantifiers
The Opposite Inference is Invalid

$$\begin{array}{l|l} 1 & \forall x\,\exists y\,\mathsf{Loves}(x,y) \\ \hline 2 & \exists y\,\forall x\,\mathsf{Loves}(x,y) \end{array}$$

- We know that this inference isn't valid
- Consider a world with two people:
  - Alice and Bob
  - Bob loves Alice
  - Alice loves Bob
  - But, Bob does not love himself
  - And Alice does not love herself
- The premise is true: everyone loves someone or other
- But the conclusion is false, no one is loved by everyone

Mixing Quantifiers

A Pseudo-Proof

$$
\begin{array}{r|l}
1 & \forall x\, \exists y\, \mathsf{Loves}(x, y) \\
\hline
2 & \exists y\, \forall x\, \mathsf{Loves}(x, y)
\end{array}
$$

Pseudo-Proof: Let b be an arbitrary boy. By premise 1, he loves some girl. Assume it's g. Since b was chosen arbitrarily, we may conclude by Univ. Intro. that $\forall x\, \mathsf{Loves}(x, g)$. The conclusion follows by exist. intro.!

- The crucial misstep: is b wasn't arbitrary!
- Why? Introducing g, a specific girl that b likes, makes b non-arbitrary!
  - Our proof then contains information particular to b: which girl he likes, namely g!

Mixing Quantifiers

The Moral

Summary

1. **Universal Introduction**: To prove $\forall x\, S(x)$, chose a new constant c and prove $S(c)$, making sure that $S(c)$ does not contain any names introduced by Exist. Elim. after the introduction of c.
   - Above, g was introduced by Exist. Elim. after b
2. Same for applications of General Conditional Proof

- If you do not follow this advice, you will be able to give 'proofs' of invalid arguments
- But, if there's proofs of invalid arguments, the whole idea of proof is bankrupt!

In-Class Exercise

Exercise 12.14

12.14 *This exercise contains a purported proof. If it is correct, say so. If it is incorrect, explain what goes wrong using the notions presented above.*

$$
\begin{array}{|l}
\hline
\exists x\, (x = x \to \neg\exists y\, x \neq y) \quad \text{(There is at most one object)}
\end{array}
$$

Purported proof: Toward a proof by contradiction, suppose $\neg\exists x\, (x = x \to \neg\exists y\, x \neq y)$. This is equivalent to $\forall x\, \neg(x = x \to \neg\exists y\, x \neq y)$, which is equivalent to $\forall x\, (x = x \land \exists y\, x \neq y)$. By Univ. Elim. we get $c = c \land \exists y\, c \neq y$. By Exist. Elim. we may assume $c \neq d$. But, since c was arbitrary, it follows that $\forall x\, x \neq d$. By Univ. Elim, we get $d \neq d$, which is a contradiction. Thus, the conclusion must be true.

Mixing Quantifiers in Proofs

Where We Are

- So far: an important lesson about how **not** to apply Univ. Intro. and Exist. Elim.
- We learned how to recognize this mistake
- But we also need to practice correctly mixing these two rules
- So let's do some more informal proofs that require mixing the two rules

Mixing Quantifiers
An Example

$$\forall x\, \forall y\, [\mathsf{Smaller}(x, y) \rightarrow \mathsf{SameShape}(x, y)]$$
$$\forall x\, \exists y\, [\mathsf{Adjoins}(x, y) \rightarrow \mathsf{Smaller}(x, y)]$$
$$\overline{\forall x\, \exists y\, [\mathsf{Adjoins}(x, y) \rightarrow \mathsf{SameShape}(x, y)]}$$

Proof: From premise 2 by Univ. Elim. $\exists y\, [\mathsf{Adjoins}(c, y) \rightarrow \mathsf{Smaller}(c, y)]$. By Exist. Elim. we may then assume $\mathsf{Adjoins}(c, d) \rightarrow \mathsf{Smaller}(c, d)$. From premise 1 by Univ. Elim. $\mathsf{Smaller}(c, d) \rightarrow \mathsf{SameShape}(c, d)$. By the transitivity of $\rightarrow$, we have $\mathsf{Adjoins}(c, d) \rightarrow \mathsf{SameShape}(c, d)$. Exist. Intro. then gives us $\exists y\, [\mathsf{Adjoins}(c, y) \rightarrow \mathsf{SameShape}(c, y)]$. Since c was arbitrary, the conclusion follows by Univ. Intro.

Mixing Quantifiers
Another Example

$$\exists y\, \forall x\, (\mathsf{Smaller}(x, y) \vee \neg \mathsf{Tet}(x))$$
$$\forall y\, \forall x\, (\mathsf{Smaller}(y, x) \rightarrow \mathsf{Cube}(y))$$
$$\forall z\, \forall y\, [(\mathsf{LeftOf}(z, y) \wedge \neg \mathsf{Tet}(z)) \rightarrow \mathsf{Cube}(z)]$$
$$\overline{\forall x\, (\forall y\, \mathsf{LeftOf}(x, y) \rightarrow \mathsf{Cube}(x))}$$

Proof: We will use general conditional proof, but first we apply Exist. Elim. to premise 1 and assume $\forall x\, (\mathsf{Smaller}(x, a) \vee \neg \mathsf{Tet}(x))$. Now we take an arbitrary c and assume $\forall y\, \mathsf{LeftOf}(c, y)$, with the goal of showing $\mathsf{Cube}(c)$. This assumption gives us $\mathsf{LeftOf}(c, a)$ by Univ. Elim. By Univ. Elim. we also have $\mathsf{Smaller}(c, a) \vee \neg \mathsf{Tet}(c)$. Consider the second case. Premise 3 gives us $(\mathsf{LeftOf}(c, a) \wedge \neg \mathsf{Tet}(c)) \rightarrow \mathsf{Cube}(c)$. Then $\mathsf{Cube}(c)$ follows by modus ponens. In the second case, premise 2 gives us $(\mathsf{Smaller}(c, a) \rightarrow \mathsf{Cube}(c))$ by Univ. Elim. So we have $\mathsf{Cube}(c)$ again by modus ponens. Thus, either way, we have $\mathsf{Cube}(c)$, and since c was arbitrary the conclusion follows by general conditional proof.