

Announcements

11.10

- ① Grades for HW1-4, HW6 and Midterm are on Bb
 - Check on them, email me with questions
 - If you've already emailed: I'm working through a backlog, be patient
- ② Midterms were returned last Thursday
 - If you missed it, come to office hours to claim it
- ③ There are 3 HWs left
 - HW is a big part of your grade
 - Make them count!

Informal Proofs with Quantifiers II

Universal Proofs

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Outline

- ① Review
- ② Universal Introduction
- ③ General Conditional Proof

Two Informal Inference Steps

Existential Introduction & Universal Elimination

Existential Introduction (Official Version)

$$\triangleright \frac{S(c)}{\exists x S(x)}$$

(When 'c' names an object in the domain of discourse)

Universal Elimination (Official Version)

$$\triangleright \frac{\forall x S(x)}{S(c)}$$

(Where 'c' refers to an object in the domain of discourse)

Two Inference Steps

A Simple Example

Example Argument

1	$\text{Tet}(a) \rightarrow \forall x \text{Small}(x)$
2	$\forall y \text{Tet}(y)$
3	$\exists x \text{Small}(x)$

Proof:

- From 2 by **universal elimination**: $\text{Tet}(a)$
- From this and 1 we get by modus ponens $\forall x \text{Small}(x)$

- Applying **universal elimination** to this, we get $\text{Small}(a)$
- By **existential introduction**: $\exists x \text{Small}(x)$ ✓

Existential Elimination

Background

- Suppose you are given an existential premise and need to use it to prove a conclusion
 - (1) *Something is a cube*
- Suppose the domain includes only two blocks a and b
- What can you infer from (1)?
 - a is a cube? **No!**
 - b is a cube? **No!**
- Here's an idea:
 - We can infer from (1) that there is some block, call it *Frank*, that is a cube
- Then we can continue on in our reasoning as if *Frank* was a real name, even though it's a dummy name (an ersatz)
- This *dummy name method* turns out to be **very** useful

Existential Elimination

In Review

The Method of Existential Elimination

- Given $\exists x S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c , and then **assume** $S(c)$
- However, c must be a **new name**, i.e. one not already in use in the context of your proof

- Remember, the whole idea of the dummy name is to remain agnostic about **which** object(s) satisfy $S(x)$
- In a proof with existential and universal premises:
 - **Always** apply existential elimination before applying universal elimination
 - This will save you space and possible confusion

Existential Elimination

An Example

Example Argument

1	$\forall z [\text{Tet}(z) \vee \text{Cube}(z)]$
2	$\forall x [\text{Tet}(x) \rightarrow \text{Small}(x)]$
3	$\exists x \neg \text{Small}(x)$
4	$\exists x \text{Cube}(x)$

Proof:

- First, we apply **existential elimination** to 3
 $\neg \text{Small}(a)$ (note 'a' is **new**)
- From 2 by **universal elimination** we get $\text{Tet}(a) \rightarrow \text{Small}(a)$
- These two facts imply that $\neg \text{Tet}(a)$
- From 1 by **universal elimination**: $\text{Tet}(a) \vee \text{Cube}(a)$
- So $\text{Cube}(a)$ must be true
- By **existential introduction**: $\exists x \text{Cube}(x)$ ✓

Summary

The Steps and Methods So Far

Method of Existential Elimination

- 1 Given $\exists xS(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c , and then **assume** $S(c)$
- 2 However, c must be a **new name**, i.e. one not already in uses in proof

Existential Introduction (Official Version)

$$\frac{S(n)}{\triangleright \exists xS(x)}$$

(When 'n' names an object in the domain of discourse)

Universal Elimination (Official Version)

$$\frac{\forall xS(x)}{\triangleright S(c)}$$

(Where 'c' refers to an object in the domain of discourse)

What We've Done

Taking Stock

- We've learned two inference steps and one proof method for quantifiers:
 - 1 Universal Elimination, Existential Introduction
 - 2 The Method of Existential Elimination
- What's missing from this list?
 - Universal Introduction
- Universal introduction is a **proof method** and requires the appeal to dummy names also
- We'll start with some example inferences

Universal Introduction

Justifying a Universal

- Suppose you are looking at Tarski's World and there are 3 blocks: a , b and c
- Suppose you are asked to prove the following **universal**:
(2) $\forall x \text{Tet}(x)$
- How might you go about it?
 - Consider each object, and show that it satisfies $\text{Tet}(x)$
 - Cumulatively, this process will justify saying that (2) is true in this world
- Call this method the *check-each-object method*

Universal Introduction

The Need for a Better Method

- Consider the fact that:
(2) $\forall x \neg[\text{Cube}(x) \wedge \text{Tet}(x)]$
This is true of **every world**
- So, we should be able to **prove** (2) without considering particular objects from a particular world
- Further, we should be able to prove it even if there were infinitely many objects
- These two facts go against the check-each-object-method:
 - That method requires you to consider particular objects from a particular world
 - It assumes that it's possible to finish checking every object
- Let's look at a more general method

Universal Introduction

An Example from Tarski's World

$$\left| \begin{array}{l} \text{---} \\ \forall x \neg[\text{Cube}(x) \wedge \text{Tet}(x)] \end{array} \right.$$

Proof: Let c be an arbitrary block. If we assume $\text{Cube}(c) \wedge \text{Tet}(c)$, then we immediately have a contradiction, since c cannot be both a cube and a tetrahedron. So it must be true that $\neg[\text{Cube}(c) \wedge \text{Tet}(c)]$. But since c was an **arbitrarily chosen** block, it must be that $\forall x \neg[\text{Cube}(x) \wedge \text{Tet}(x)]$.

- The key in this proof is the use of a **dummy name** to talk about an arbitrary block

Universal Introduction

An Example from the Real World

Anyone who passes Phil 2310 with an A is smart
 Every math major has passed Phil 2310 with an A

 Every math major has been smart

Proof: Let 'Jessica' refer to any one of the math majors. By the second premise, Jessica must have passed Phil 2310 with an A (universal elimination). Then by the first premise, Jessica must have been smart. But since Jessica was an **arbitrarily chosen** math major, it follows that **every** math major was smart.

- The key in this proof is the use of a dummy name to talk about an arbitrary math major

Universal Introduction

The Important Features of Our Proof

- Notice in our proofs we didn't need to consider a particular set of blocks or math majors
- Our proof method was perfectly general: it works regardless of which set of objects you apply it to
- This generality was achieved by introducing a new name to talk about an arbitrary object
- That object was an arbitrary representative
 - We inferred something about that object
 - So the same must follow for every object
- This is the basic idea behind **Universal Introduction**

Universal Introduction

The Official Formulation

Universal Introduction

To prove $\forall x S(x)$:

- 1 Introduce a **new** name c to stand for a completely **arbitrary** member of the domain of discourse
- 2 Prove $S(c)$
- 3 Conclude $\forall x S(x)$

- Since c is arbitrary, showing $S(c)$ shows $\forall x S(x)$
- c 's being arbitrary **prevents** one from assuming that any properties **specific to one object** are used in proof

Universal Introduction

Another Example

Example Argument

1	$\forall x \text{Tet}(x)$
2	$\forall x \text{Medium}(x)$
3	$\forall x (\text{Tet}(x) \wedge \text{Medium}(x))$

Proof:

- Let 'c' be an arbitrary block
- From 1 Tet(c) follows by universal elimination
- Applying universal elimination to 2 gives us Medium(c)

- So we have $\text{Tet}(c) \wedge \text{Medium}(c)$
- But c was arbitrary, so it follows that $\forall x (\text{Tet}(x) \wedge \text{Medium}(x))$ ✓

Universal Introduction

In Class Exercise

Give an informal proof for:

1	$\forall y (y \neq b \rightarrow \text{LeftOf}(y, b))$
2	$\forall x [\text{LeftOf}(x, b) \rightarrow \text{SameSize}(x, a)]$
3	$\forall x \exists y \text{SameSize}(x, y)$

Hint: use universal introduction. Premise 2 says *Every block left of b is the same size as a*. The conclusion says that *Every block is the same size as some block or other*.

General Conditional Proof

How to Prove a Universal Conditional

- In practice, we are usually concerned with proving universal claims of these forms:
 - *Every A is B*
 - *All A are B*, etc.
- As we all know, these are translated in FOL as:

$$\forall x (A(x) \rightarrow B(x))$$

- To prove this using universal introduction you would prove, for an arbitrary c:

$$A(c) \rightarrow B(c)$$

- This would be achieved using **conditional proof**:
 - Assume A(c) and show B(c)

Conditional Proof

Review of Conditional Proof

The Method of Conditional Proof

To prove $P \rightarrow Q$, temporarily assume P. If you can show Q with this additional assumption, you can infer $P \rightarrow Q$

Truth Table for \rightarrow

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- The only way for $P \rightarrow Q$ to be F is for P to be true and Q be F
- So, if you can show that when P is T Q is also T, you've shown that $P \rightarrow Q$ cannot be false, i.e. that it is true!

Conditional Proof

Review of Conditional Proof: An Example

Let's use conditional proof and modus ponens to give a proof of:

ARGUMENT 1

Tet(a) → Tet(b)	
Tet(b) → Tet(c)	
Tet(a) → Tet(c)	

Our goal is a conditional, so we use conditional proof.

Proof: Suppose Tet(a). Then by premise 1 Tet(b) follows by modus ponens. But then we may now again use modus ponens and premise 2 to infer Tet(c). This is the consequent of our goal, so we have successfully completed our conditional proof.

Universal Introduction + Conditional Proof

How to Prove a Universal Conditional

$\forall x \text{ Cube}(x)$	
$\forall x (\text{Small}(x) \rightarrow \text{Cube}(x))$	

Proof: Let c be an arbitrary block. We will show $\text{Small}(c) \rightarrow \text{Cube}(c)$ by conditional proof. Suppose $\text{Small}(c)$. By Premise 1 and Univ. Elim. $\text{Cube}(c)$. Thus, the conditional follows. c was arbitrary so conclusion follows by Univ. Intro.

Important Observation

Proving a universal conditional combines two proof methods:

- 1 Introducing an arbitrary constant, c
- 2 Assuming the antecedent holds for it,
- 3 Showing that the conclusion does too.

General Conditional Proof

Universal Instantiation Plus Conditional Proof

- Proofs will often involve using conditional proof & universal introduction together
- So, let's introduce a short-cut & name for it

General Conditional Proof

To prove $\forall x (A(x) \rightarrow B(x))$:

- 1 Introduce a **new** name c to stand for a completely **arbitrary** member of the domain of discourse
- 2 Assume $A(c)$
- 3 Prove $B(c)$
- 4 Conclude $\forall x (A(x) \rightarrow B(x))$

- This is equivalent to using universal introduction along with conditional proof

General Conditional Proof

An Example

Proof:

- Let a name an arbitrary block
- Suppose $\text{Small}(a)$
(Goal: Show $\text{Cube}(a)$)
- From premise 1:
 $\text{Small}(a) \rightarrow \neg \text{Tet}(a)$
- By modus ponens, we get $\neg \text{Tet}(a)$
- Premise 2 gives us $\neg \text{Tet}(a) \rightarrow \text{Cube}(a)$, so by modus ponens we have $\text{Cube}(a)$, (our goal)
- Since a was arbitrary, it follows that
 $\forall x [\text{Small}(x) \rightarrow \text{Cube}(x)] \quad \checkmark$

Example Argument

$\forall x [(\text{Small}(x) \rightarrow \neg \text{Tet}(x))$	
$\forall x [\neg \text{Tet}(x) \rightarrow \text{Cube}(x)]$	
$\forall x [\text{Small}(x) \rightarrow \text{Cube}(x)]$	

General Conditional Proof

Another Example

Proof:

- Let a name an arbitrary block
- Suppose $\text{Medium}(a)$
(Goal: Show $\neg\text{Smaller}(a, c)$)
- From premise 1:
 $(\text{Cube}(a) \wedge \text{Large}(a)) \vee (\text{Medium}(a) \wedge \text{Tet}(a))$
 - Since $\text{Medium}(a)$, the first disjunct must be false, and $\text{Medium}(a) \wedge \text{Tet}(a)$ must be true
 - Premise 2 gives us $\text{Tet}(a) \rightarrow \neg\text{Smaller}(a, c)$, so by modus ponens we have $\neg\text{Smaller}(a, c)$, (our goal)
 - Since a was arbitrary, it follows that $\forall x [\text{Medium}(x) \rightarrow \neg\text{Smaller}(x, c)]$ ✓

Example Argument

$$\begin{array}{l} \forall x [(\text{Cube}(x) \vee \text{Large}(x)) \\ \quad \vee (\text{Medium}(x) \wedge \text{Tet}(x))] \\ \forall x [\text{Tet}(x) \rightarrow \neg\text{Smaller}(x, c)] \\ \hline \forall x [\text{Medium}(x) \rightarrow \neg\text{Smaller}(x, c)] \end{array}$$

General Conditional Proof

In Class Exercise

Give an informal proof for:

$$\begin{array}{l} 1 \quad \forall y [\exists x \text{Tet}(x) \rightarrow \text{LeftOf}(y, b)] \\ 2 \quad \forall x [\text{LeftOf}(x, b) \rightarrow \text{Smaller}(x, a)] \\ 3 \quad \forall x [\text{Tet}(x) \rightarrow \text{Smaller}(x, a)] \end{array}$$

Hint: use the method of general conditional proof, along with universal elimination, existential introduction and modus ponens.

Universal Proof

Summary

Summary

- To prove a universally quantified claim, use **Universal Introduction**
 - E.g. to prove $\forall x \text{Tet}(x)$, use Univ. Intro.
 - When proving a universal conditional, you may use **General Conditional Proof**
 - This is just Univ. Intro. together with Conditional Proof
 - These are both **proof methods**
- Next class, we will learn how to mix Univ. Intro. with the method of Existential Elimination