Announcements
11.08

1. Midterm grades are on Bb
   - Midterms will be handed back at the end of class
2. Other grades are slowly showing up on Bb

Informal Proofs with Quantifiers I
Inference Steps and Existential Instantiation

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Outline

Shifting Gears
Proof

- We have learned about what quantifiers mean
- Now it is time to think about how quantificational sentences should be used in proofs
- We have not done any proofs for a while, so let's remind ourselves of what they are all about
- Proofs are step-by-step demonstrations of a conclusion from some premises
  - Each step is justified by the meanings of the terms involved
- Proofs can be formal or informal

Inference Steps

What They Are

- An inference step is a simple transition from one claim C_1 to another C_2
- Valid inference steps are ones where the truth of C_1 guarantees the truth of C_2
- So far, we have not learned which steps involving quantifiers are valid
- In this section of the lecture we are going to learn two important inference steps for quantifiers

Universal Elimination

An Example

- Suppose you are really convinced of this generalization:

  (1) *Everyone has DNA*
- Now consider a random person: Michael Jackson
- Given (1), what can we infer about MJ?

  (2) *MJ has DNA*
- Why?
- (2) logically follows from (1)!
  - If (1) is true it is impossible for (2) to be false!

Universal Elimination

The Inference Pattern at Work

(1) *Everyone has DNA*

(2) *Michael Jackson has DNA*

- (2) is a logical consequence of (1)
- But this is just one example of more general valid inference pattern:

Universal Elimination (Unofficial Version)

| Everything is an F
|---
▷ | c is an F

(Where 'c' is a name for an actually existing object)

Universal Elimination

The Official Version

- Our unofficial version of universal elimination gets the basic idea right
- But, it's not quite as general as it should be
- To make it more general, it is helpful to write it in terms of FOL:

Universal Elimination (Official Version)

From $\forall x\, S(x)$ you may infer $S(c)$, as long as 'c' refers to an object in the domain of discourse.

- This is an informal inference step

Universal Elimination

Another Example

Universal Elimination (Official Version)

From $\forall x\, S(x)$ you may infer $S(c)$, as long as 'c' refers to an object in the domain of discourse.

- Suppose you are given:

  (3) $\forall x\,(\mathsf{Cube}(x) \lor \mathsf{Small}(x))$

  And you also know that a and b name objects in the domain of discourse
- Then you can infer, by universal elimination:

  (4) $\mathsf{Cube}(a) \lor \mathsf{Small}(a)$

  (5) $\mathsf{Cube}(b) \lor \mathsf{Small}(b)$

Existential Introduction

An Example

(6) *George Bush greeted the Pope*

- So:

  (7) *Someone met the pope*

(8) *Everyone loves Plato*

- So:

  (9) *Everyone loves someone*

Existential Introduction (Unofficial Version)

| c is an F
|---
▷ | Something is an F

Existential Introduction

The Official Version

- Again, this informal rule is clearer and more general when stated using FOL:

Existential Introduction (Official Version)

From $S(c)$ you may infer $\exists x\, S(x)$, as long as 'c' refers to an object in the domain of discourse.

- Example:

  (10) $\mathsf{Tet}(a) \lor \neg\mathsf{SameSize}(a, c)$

  Therefore, by existential introduction:

  (11) $\exists x\,(\mathsf{Tet}(x) \lor \neg\mathsf{SameSize}(x, c))$

Existential Introduction

Why Bother Mentioning the Domain of Discourse?

Existential Introduction (Official Version)

From $S(c)$ you may infer $\exists x\, S(x)$, as long as 'c' refers to an object in the domain of discourse.

- So from:

  (12) *Santa Claus does not exist*

  Does it follow by existential introduction that:

  (13) There exists an x that does not exist
- No! *Santa Claus* does not name an object in the domain of discourse

An Example Proof

Putting Together our Two Inference Steps

Example Argument

1	$\forall x\,[\mathsf{Tet}(x) \to \mathsf{Small}(x)]$
2	$\mathsf{Tet}(a)$
3	$\exists x\,[\mathsf{Tet}(x) \land \mathsf{Small}(x)]$

Proof:

- From 1 by universal elimination we get $\mathsf{Tet}(a) \to \mathsf{Small}(a)$
- From this and 2 we get by modus ponens $\mathsf{Small}(a)$

- So we have $\mathsf{Tet}(a) \land \mathsf{Small}(a)$
- By existential introduction it follows that: $\exists x\,[\mathsf{Tet}(x) \land \mathsf{Small}(x)]$ ✓

In Class Exercise

An Informal Proof

The Exercise

Give an informal proof that this argument is valid

1	$\forall x\,[\mathsf{Tet}(x) \lor \mathsf{Small}(x)]$
2	$\neg\mathsf{Tet}(a)$
3	$\exists x\,\mathsf{Small}(x)$

The Inference Steps

1. **Universal Elimination**: from $\forall x\,\mathsf{S}(x)$ you can infer $\mathsf{S}(c)$, as long as c names an object in the domain
2. **Existential Introduction**: from $\mathsf{S}(c)$ you can infer $\exists x\,\mathsf{S}(x)$, as long as c names an object in the domain

Summary

Two Inference Steps

Summary

- For the quantifiers $\forall$ and $\exists$ there are two informal valid inference steps:
  1. **Universal Elimination**: from $\forall x\,\mathsf{S}(x)$ you can infer $\mathsf{S}(c)$, as long as c names an object in the domain
  2. **Existential Introduction**: from $\mathsf{S}(c)$ you can infer $\exists x\,\mathsf{S}(x)$, as long as c names an object in the domain
- There are two other, more involved, methods of proof for the quantifiers

Today, we'll learn one of them: existential elimination

Existential Elimination

Background

- Suppose you are given an existential premise and need to use it to prove a conclusion

  (14) *Something is either a cube or not small*
- Suppose the domain includes only two blocks a and b
- What can you infer from (14)?
  - a is a cube or not small? No!
  - b is a cube or not small? No!
- Here's an idea:
  - We can infer from (14) that there is some block, call it *Frank*, that is either a cube or not small
- Then we can continue our reasoning as if *Frank* was a real name, even though it's a dummy name
- This dummy name method turns out to be **very** useful

Existential Elimination
An Example

Example Argument

1	$\forall x\,[\mathsf{Tet}(x) \rightarrow \mathsf{Small}(x)]$
2	$\exists x\,\mathsf{Tet}(x)$
3	$\exists x\,\mathsf{Small}(x)$

Proof:

- We need to use 2; let's try the dummy name method
- From 2 we know there is some block, call it d, such that $\mathsf{Tet}(d)$
- From 1 by universal elimination we get $\mathsf{Tet}(d) \rightarrow \mathsf{Small}(d)$
- So we have $\mathsf{Small}(d)$ by modus ponens
- By existential introduction it follows that: $\exists x\,\mathsf{Small}(x)$ ✓

Existential Elimination
An Observation

Example Argument

1	$\forall x\,[\mathsf{Tet}(x) \rightarrow \mathsf{Small}(x)]$
2	$\exists x\,\mathsf{Tet}(x)$
3	$\exists x\,\mathsf{Small}(x)$

Observation:

- In our proof of this argument we introduced our dummy name and **then** used universal elimination
- Would opposite order work?
- Suppose from 1 by univ. slim: $\mathsf{Tet}(d) \rightarrow \mathsf{Small}(d)$
- Can we use dummy name method: *let d be whatever is a tet by 2*
- No! The essence of the dummy name method is to introduce a new name, but d is already in use here
- If we pick a different dummy name, say e, we get nowhere, unless we use univ. elim. all over again to get $\mathsf{Tet}(e) \rightarrow \mathsf{Small}(e)$

Existential Elimination
Our Observation

- To summarize:
  - Always apply universal elimination after invoking the dummy name method
- Better name & description for 'dummy name method':

Method of Existential Elimination

1. Given $\exists x\,\mathsf{S}(x)$, you may give a dummy name to (one of) the object(s) satisfying $\mathsf{S}(x)$, say c, and then assume $\mathsf{S}(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

Existential Elimination
Official Formulation

Method of Existential Elimination

1. Given $\exists x\,\mathsf{S}(x)$, you may give a dummy name to (one of) the object(s) satisfying $\mathsf{S}(x)$, say c, and then assume $\mathsf{S}(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

- Remember, the whole idea of the dummy name is to remain agnostic about what object(s) satisfy $\mathsf{S}(x)$
- Using an old name would violate this agnosticism
  - Old names are real names
  - And real names name particular objects; dummy names don't

Summary
Existential Elimination

Summary

1. Existential elimination is a method of proof
2. It's a tool for using $\exists x\, S(x)$ in further reasoning:
   - It allows you to talk about the thing that satisfies $S(x)$ by giving it a temporary name
   - But keep in mind that this must be a new name since $\exists x\, S(x)$ does not allow you to infer which particular thing satisfies $S(x)$
3. When doing a proof with universal and existential premises, always use existential elimination before universal elimination

Existential Elimination
Another Example

$\forall y\, [\text{Cube}(y) \lor \text{Dodec}(y)]$

$\forall x\, [\text{Cube}(x) \rightarrow \text{Large}(x)]$

$\exists x\, \neg\text{Large}(x)$

$\exists x\, \text{Dodec}(x)$

Proof: From premise 3 by exist. elim. we may assume $\neg\text{Large}(b)$. From premise 2 by univ. elim. we know that $\text{Cube}(b) \rightarrow \text{Large}(b)$. So, it must be that $\neg\text{Cube}(b)$. But, from premise 1 by univ. elim. we get $\text{Cube}(b) \lor \text{Dodec}(b)$, so it follows that $\text{Dodec}(b)$. From this we can get to our desired conclusion by existential introduction: $\exists x\, \text{Dodec}(x)$.

In Class Exercise

Give an informal proof that the following argument is valid:

1 $\forall x\, [\text{Tet}(x) \lor \neg\text{Small}(x)]$

2 $\forall y\, [\text{Tet}(y) \rightarrow \text{LeftOf}(a, y)]$

3 $\exists x\, \text{Small}(x)$

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4 $\exists x\, \text{LeftOf}(a, x)$

You may use any of the proof methods or inference steps discussed so far in this class

Summary
The Steps and Methods from Today

Method of Existential Elimination

1. Given $\exists x\, S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c, and then assume $S(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

Existential Introduction (Official Version)

From $S(c)$ you may infer $\exists x\, S(x)$, as long as 'c' refers to an object in the domain of discourse.

Universal Elimination (Official Version)

From $\forall x\, S(x)$ you may infer $S(c)$, as long as 'c' refers to an object in the domain of discourse.