Multiple & Mixed Quantifiers

Understanding Quantification

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11.01.11

Quantification

What We've Done

1. So far, we've learned what $\forall$ and $\exists$ mean
   - Recall the **semantics** and **game rules**
   - Both based onsatisfaction
2. Use $\forall$ and $\exists$ for translation of quantifiers
   - Remember the four Aristotelian Forms
3. Two logical concepts
   - FO Validity
     - Logical truth restricted to $\forall, \exists, =, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$
   - FO Consequence
     - Logical consequence restricted to $\forall, \exists, =, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$
   - We test for these using the replacement method

Quantification

We are Just Getting Started

- This is a good start, but there is a lot more to understanding the logic of quantifiers
- Today we are going to think about what sentences containing multiple quantifiers mean
- As well as how to translate them into FOL
- We've only looked at sentences w/1 quantifier:
  - *All basketballs are orange*
  - *Some ninjas are not sociable*
- But what happens when there are 2, 3 or 4 quantifiers?

Quantification

Multiple Quantifiers

- Recall what old Abe said:

  > *You may fool all of the people some of the time; you can even fool some of the people all of the time; but you can't fool all of the people all of the time*

- Count the quantifiers: 6!
- The point:
  - We often communicate logically interesting things with several quantifiers
- So, as students of logic, we need learn how to mix multiple quantifiers

Multiple Existentials

A Simple Example

- We will begin by considering sentences with multiple occurrences of one quantifier
  (1) *Some cube is left of some tetrahedron*
- How should we represent (1) in FOL?
- We have many options
- Let's consider and compare them

Multiple Existentials

Translating our Simple Example

(1) *Some cube is left of some tetrahedron*

- Two (of the many) correct translations:
  (1a) $\exists x \exists y\, [\mathsf{Cube}(x) \land \mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y)]$
  - There are objects x and y such that: x is a cube, y is a tetrahedron and x is left of y

  (1b) $\exists x\, [\mathsf{Cube}(x) \land \exists y\, (\mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y))]$
  - There is an object x such that x is a cube and there exists an object y such that y is a tetrahedron and x is left of y
- (1a) stacks all of the quantifiers at the beginning
  - This makes it easier to paraphrase
  - But less like the English (1)!

Multiple Existentials

Multiplicity of Translations

- In addition to:
  (1a) $\exists x \exists y\, [\mathsf{Cube}(x) \land \mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y)]$
  (1b) $\exists x\, [\mathsf{Cube}(x) \land \exists y\, (\mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y))]$
  - We can put things in the reverse order:
    (1c) $\exists y \exists x\, [\mathsf{Cube}(x) \land \mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y)]$
    (1d) $\exists y\, [\mathsf{Tet}(y) \land \exists x\, (\mathsf{Cube}(x) \land \mathsf{LeftOf}(x, y))]$
  - Or put the predicates in a different order:
    (1e) $\exists x \exists y\, [\mathsf{Tet}(y) \land \mathsf{Cube}(x) \land \mathsf{LeftOf}(x, y)]$
    (1f) $\exists x\, [\mathsf{Cube}(x) \land \exists y\, (\mathsf{LeftOf}(x, y) \land \mathsf{Tet}(y))]$
- Let's look at these in Tarski's World to see that they are equivalent (`Equivalences.sen /.wld`)

Translation Convention

A Helpful Note

Translation Conventions (Stylistic Advice)

1. All quantifiers are stacked up 'out in front'
2. 1st quantifier in English sentence is written 1st and binds x, 2nd goes 2nd and binds y, etc.
3. List predicates in order of quantifiers they restrict
   - Translate: *some cube is left of some tetrahedron*
     (1a) $\exists x \exists y\, [\mathsf{Cube}(x) \land \mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y)]$
   - Rather than:
     (1e) $\exists x \exists y\, [\mathsf{Tet}(y) \land \mathsf{Cube}(x) \land \mathsf{LeftOf}(x, y)]$
   - $\mathsf{Cube}(x)$ goes before $\mathsf{Tet}(y)$ since $\exists x$ come before $\exists y$, $\mathsf{Left}(x, y)$ goes last since it restricts neither $\exists x$ nor $\exists y$

Translation

Comments on Our Convention

- In general, there are very many different but equally correct ways of translating quantified sentences
  - Especially in sentences with multiple quantifiers
  - By equally correct we mean FO Equivalent
- Conventions on previous slide are **sylistic**
- Prenex Form: all of a formula's quantifiers are stacked up at the front of the formula
  - Like: $\exists x \exists y\, (\mathsf{Cube}(x) \land \mathsf{Tet}(y))$
  - Not: $\exists y\, (\mathsf{Cube}(x) \land \exists y\, \mathsf{Tet}(y))$
- Everything we've said so far also holds for sentences containing multiple universal quantifiers

Multiple Universals

(2) *Every tetrahedron is larger than every cube*

- Given our conventions, the natural translation is:

  (2a) $\forall x \forall y\, [(\mathsf{Tet}(x) \land \mathsf{Cube}(y)) \rightarrow \mathsf{Larger}(x, y)]$

  - For every block x and every block y, if x is a tetrahedron and y is a cube then x is larger than y
- But this is equivalent to (among others):

  (2b) $\forall x\, [\mathsf{Tet}(x) \rightarrow \forall y\, (\mathsf{Cube}(y) \rightarrow \mathsf{Larger}(x, y))]$
- Let's look at Tarski's World (`Equivalences.sen / .wld`)

Multiple Quantifiers

An Important Fact

Fact 1 (Multiplied Quantifiers)

When you have multiple occurrences of a single quantifier, order does not matter:

1. $\exists x \exists y\, P(x, y) \Leftrightarrow \exists y \exists x\, P(x, y)$
2. $\forall x \forall y\, P(x, y) \Leftrightarrow \forall y \forall x\, P(x, y)$

A Tricky Fact

Resisting the Temptation...

- It is tempting to paraphrase:

  (3) $\forall x \forall y\, [(\mathsf{Small}(x) \land \mathsf{Cube}(y)) \rightarrow \mathsf{RightOf}(x, y)]$

  As:

  (4) For every block x and every **other** block y, if x is small and y is a cube then x is right of y
- But RESIST!
  - (4) is not what (3) means
- (4) is really a paraphrase of:

  (5) $\forall x \forall y\, [(x \neq y \land \mathsf{Small}(x) \land \mathsf{Cube}(y)) \rightarrow \mathsf{RightOf}(x, y)]$
- (3) and (5) are not equivalent
- See this in TW (`Identity.sen, Identity.wld`)

The Tricky Fact

The Moral of the Story

The Tricky Fact

1. When evaluating sentences with multiple quantifiers, don't fall into the trap of thinking that distinct variables range over distinct objects
2. In fact, $\forall x \, \forall y \, P(x, y)$ logically entails $\forall x \, P(x, x)$, so the variables can't be assumed to range over distinct variables. (The same goes for $\exists$)

Mixing Quantifiers

Doing Things Differently

- In addition to repeating the same quantifier, you can mix quantifiers:

  (6) *Everyone loves someone or other*

  (7) *There is someone that everyone loves*
- Both (6) and (7) mix a universal and an existential
- But, they do it differently:
  - (6) is a *Universal Existential*
  - (7) is an *Existential Universal*
- Accordingly, we translate (6) and (7) differently:

  (6′) $\forall x \, \exists y \, (\mathsf{Love}(x, y))$

  (7′) $\exists y \, \forall x \, (\mathsf{Love}(x, y))$

Mixing Quantifiers

The Difference in Meaning is Big

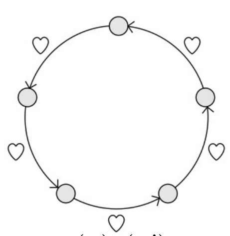
(6) *Everyone loves someone or other*

(6′) $\forall x \, \exists y \, (\mathsf{Love}(x, y))$

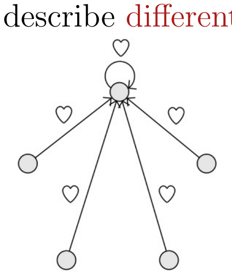
(7) *There is someone that everyone loves*

(7′) $\exists y \, \forall x \, (\mathsf{Love}(x, y))$

- (6)/(6′) and (7)/(7′) describe different situations:



(6),(6′)



(7),(7′)

Mixing Quantifiers

Entailment Relations

(6) *Everyone loves someone or other*

(6′) $\forall x \, \exists y \, (\mathsf{Love}(x, y))$

(7) *There is someone that everyone loves*

(7′) $\exists y \, \forall x \, (\mathsf{Love}(x, y))$

Fact

(7) entails (6). By (7) there's some person, call him/her Pat, that everyone loves. It follows that everyone loves someone (or other), namely Pat!

Fact

(6) does not entail (7). Everyone could love a different person. Then (6) is true but (7) is not

Mixing Quantifiers

The Important Difference

- What examples (6) and (7) show is that when you mix quantifiers order does matter!
- This is very different from multiple occurrences of a single quantifier:
  - In that case, order does not matter
- To solidify the difference between *existential-universal* and *universal-existential* let's look at some examples in Tarski's World (`MQ World.wld, MQ World 2.wld, MQ Sentences.sen`)

Summary

Two Facts

Fact 1 (Multiplied Quantifiers)

When you have multiple occurrences of a single quantifier, order does not matter:

1. $\exists x\, \exists y\, P(x, y) \Leftrightarrow \exists y\, \exists x\, P(x, y)$
2. $\forall x\, \forall y\, P(x, y) \Leftrightarrow \forall y\, \forall x\, P(x, y)$

Fact 2 (Mixed Quantifiers)

When you have multiple occurrences of different quantifiers, order does matter:

- $\forall x\, \exists y\, P(x, y) \not\Leftrightarrow \exists y\, \forall x\, P(x, y)$

Exercise

Mixed Quantifiers in Tarski's World

11.11 (Building a world) Create a world in which all ten sentences in Arnault's Sentences are true.