

## Announcements

10.27

# The Logic of Quantifiers

## Logical Truth & Consequence in Full FOL

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- ① Hang tight on the midterm
  - We'll get it back to you as soon as we can
- ② Grades for returned HW will be posted to Bb soon

## Outline

- ① Introduction
- ② FO Validity
- ③ FO Consequence

# Overview

## The Big Picture

- Now that we've added  $\forall$  and  $\exists$ , we have introduced every connective of FOL:

$$\forall \quad \exists \quad \leftrightarrow \quad \rightarrow \quad \vee \quad \wedge \quad \neg \quad =$$

- For six of these symbols we've studied:
  - ① It's semantics: truth-tables, satisfaction, game rules
  - ② How to translate English sentences using it
  - ③ It's role in logic: which sentences are logical truths and which arguments are valid
  - ④ It's role in proofs: which inference steps and methods of proof it supports and how these can be formalized
- For  $\forall$  and  $\exists$ , we've only done the first two
- Today, we'll get started on the third!

# Overview

Today

- So today we'll be interested in two questions:
  - Which quantificational sentences are logical truths?
  - Which arguments containing quantifiers are valid?
- We'll start by reviewing our past discussion of logical truths and logical consequence

# The Logical Concepts

Logical Truth & Logical Consequence

## Logical Truth

A is a **logical truth** iff it is **impossible** for A to be false given the meaning of the logical vocabulary it contains

## Logical Consequence

C is a **logical consequence** of  $P_1, \dots, P_n$  iff it is **impossible** for  $P_1, \dots, P_n$  to be true while C is false

- Both of these concepts are at the very heart of logic
  - But, they are annoyingly vague and imprecise
  - What exactly is meant by *impossible*?
- In the first half of the class we explored one method for making logical possibility precise: **truth tables**

# Truth Tables

Their Spoils

- Truth tables allowed us to define these concepts:
  - Tautology**  
A is a tautology iff every row the truth table assigns T to A
  - Tautological Consequence**  
C is a tautological consequence of  $P_1, \dots, P_n$  iff every row of their joint truth table which assigns T to  $P_1, \dots, P_n$  also assigns T to C

# Truth Tables

Their Drawbacks

- These definitions are a step towards better understanding logical truth and consequence:
  - Every tautology is an (intuitive) logical truth
  - Every tautological consequence is an (intuitive) logical consequence
- But the step is **not** complete:
  - Some logical truths are not tautologies
  - Some logical consequences are not tautological consequences
- The difficulty was that the notion of logical possibility used in truth tables was not discerning enough

# Truth Tables

Not Discerning Enough

- Recall the procedure for building a truth-table:

- 1 Build ref. col's
- 2 Fill ref. col's
- 3 Fill col's under connectives

Truth Table		
$a = a$	$b = b$	$a = a \wedge b = b$
T	T	T
T	F	F
F	T	F
F	F	F

- This table shows that  $a = a \wedge b = b$  is **not** a tautology: there are some F's in the main column
- But it **can't be false**; it's a logical truth!

# Truth Tables

Not Discerning Enough

Truth Table		
$a = a$	$b = b$	$a = a \wedge b = b$
T	T	T
T	F	F
F	T	F
F	F	F

- In building truth tables, possibilities are included which are not genuine logical possibilities
  - It is not logically possible for  $a = a$  or  $b = b$  to be F!

# Discussion

Truth Tables & Logical Possibility

- The same deficiency causes there to be logical consequences which are not tautological consequences
  - Example:  $a = c$  is a logical but not a tautological consequence of  $a = b \wedge b = c$
- Why not just leave rows out if they aren't genuine logical possibilities?
- This robs truth tables of their purpose:
  - They were supposed to be a precise way of analyzing logical possibility
  - If we just appeal to intuitions about logical possibility in building table, our analysis gets us nowhere
- We want a better analysis of logical possibility!

# Tying In Quantification

We Need That Better Analysis Even More

- In case you weren't already convinced that truth tables left something to be desired, think about how few of the quantificational logical truths are tautologies
  - $\forall x (\text{Cube}(x) \rightarrow \text{Cube}(x))$  (Not a Tautology)
  - $\forall x (\text{Cube}(x) \vee \neg \text{Cube}(x))$  (Not a Tautology)
  - $\exists x (x = x)$  (Not a Tautology)
- Although some logical truths with quantifiers are tautologies:
  - $\forall x \text{Cube}(x) \vee \neg \forall x \text{Cube}(x)$  (Tautology)
  - $\neg(\exists x \text{Cube}(x) \wedge \neg \exists x \text{Cube}(x))$  (Tautology)

# FO Validity

## A Small Step

### Logical Truth

A is a **logical truth** iff it is **impossible** for A to be false given the meaning of the logical vocabulary it contains

- We are only interested in  $\forall, \exists, \leftrightarrow, \rightarrow, \vee, \wedge, \neg$  and  $=$ , so we are interested in a more limited concept

### First-Order Validity (FO Validity)

A **sentence** A is a **first-order validity** just in case it is impossible for A to be false, given the meanings of  $\forall, \exists, \leftrightarrow, \rightarrow, \vee, \wedge, \neg$  and  $=$

- Better named *First-Order Logical Truth*

# FO Validity

## An Idea

- We need to be more clear about the notion of logical possibility used to define FO validity
- Here's the insight we'll build on
- The FO validities are sentences which are true purely in virtue of the meaning of  $\forall, \exists, \leftrightarrow, \rightarrow, \vee, \wedge, \neg$  and  $=$
- If their truth comes solely from logical symbols, then you should be able ignore meaning of its predicates (except  $=$ ) and names and still get a true sentence
- Any variation of the meaning of the non-logical symbols is a logical possibility

# An Example

## Use a Non-Sense Predicate

(1)  $\forall x (\text{Cube}(x) \rightarrow \text{Cube}(x))$

- It sounds true even with a non-sense predicate:
  - (2)  $\forall x (\text{Blornk}(x) \rightarrow \text{Blornk}(x))$
  - (3) *All blornks are blornks*
- There's no interpretation of 'Blornk' according to which (2) isn't true
- So (1) remains true no matter how we interpret its non-logical symbols
- So (1) must be a FO validity

# Another Example

## Use a Non-Sense Predicate

(4)  $\forall x \text{Rich}(x) \rightarrow \text{Rich}(\text{mc.hammer})$

- Clearly true even w/ non-sense predicates and names:
  - (5)  $\forall x \text{Rorg}(x) \rightarrow \text{Rorg}(\text{dude})$
  - (6) *If everything is a rorg, then dude is a rorg*
- So (4) must be a FO validity

## Yet Another Example

Use a Non-Sense Predicate

(7)  $\neg\exists x \text{LeftOf}(x, x)$

- Replace meaningful predicate with meaningless one:

(8)  $\neg\exists x \text{Glirs}(x, x)$

- Is this obviously true?
- No, depends on whether something can glir itself
  - What if glirring is seeing?
- So the the truth of (7) is a not a fact about the meaning of **logical symbols**, so it is **not** a FO validity

## The Replacement Method

Finding FO Validities and Counterexamples

(7)  $\neg\exists x \text{LeftOf}(x, x)$

- We saw that, intuitively, (7) is not a logical truth
- We want to have a more precise way of showing this

### The Replacement Method (Basic Idea)

- 1 Replace predicates and names with non-sense names when checking for FO validity
- 2 Then consider whether or not there is any **reinterpretations** of the formula that falsify it
- 3 If there are, specify such an interpretation
  - This specification is called a **counterexample**
- 4 If there aren't, then the formula is a logical truth

## The Replacement Method

Formulating a Counterexample

(7)  $\neg\exists x \text{LeftOf}(x, x)$

### Creating a Counterexample to (7)

- 1 Replace predicates & names w/non-sense ones:
 

(8)  $\neg\exists x \text{Glirs}(x, x)$
- 2 Try to reinterpret the non-sense and make the reinterpreted formula false:
  - Let Glirs mean *loves*
  - As a matter of fact  $\text{Loves}(\text{tom.cruise}, \text{tom.cruise})$
  - In this case  $\neg\exists x \text{Loves}(x, x)$  is false
  - Therefore (7) is not a logical truth!

## FO Validity

The Replacement Method for FO Validities

### The Replacement Method (FO Validities)

The following method can be used to check whether or not S is a FO Validity

- 1 Systematically replace all of the predicates, except =, and names with meaningless predicates and names
- 2 Try to formulate a circumstance and interpretation of the nonsense in which S is false.
  - If there is **no** such circumstance and interpretation, S is a FO validity
  - If there **is** such a circumstance and interpretation, it's called a **counterexample** and S is not a FO validity

## FO Validty

## One More Example

(9)  $\forall x (\text{Larger}(x, a) \rightarrow \text{Smaller}(a, x))$

- 1 Replace predicates and names with non-sense:

(9')  $\forall x (\text{Lirrs}(x, \text{alf}) \rightarrow \text{Stams}(\text{alf}, x))$

- 2 Try to assign a meaning to the non-sense and construct a circumstance in which (9') is false:
  - Let Lirrs mean *dates* and Stams mean *likes*
  - Consider the following circumstance: Alf dates Bea, but Alf doesn't like her
  - So  $\neg(\text{Lirrs}(\text{bea}, \text{alf}) \rightarrow \text{Stams}(\text{alf}, \text{bea}))$
  - Thus,  $\forall x (\text{Lirrs}(x, \text{alf}) \rightarrow \text{Stams}(\text{alf}, x))$  is false
- So (9) is **not** a logical truth

## FO Validity

## Fitch

- Fitch also provides a tool for studying FO Validities (FO Logical Truths)

## FO Con

- **FO Con** is like **Ana Con**, except it looks only at the meanings of the **logical symbols**
- You can test if a sentence is a FO Validity by seeing if it follows from no premises using **FO Con**
- Let's look at this in Fitch (Exercise 10.24)

## The Replacement Method

## Discussion

- The replacement method is nice and all, but it doesn't seem very precise
- We just search for interpretations and circumstances and if **we** can't do it, it's a logical truth?
  - No. There is an objective fact of the matter about whether or not it can be done
- Although this search seems hazy and unstructured, it can be made much more precise
  - This would involve learning a branch of mathematics called *model theory*, which is beyond our aspirations in this class
  - Chapter 18 of *LPL* uses model theory to make the replacement method more precise

## The Replacement Method

## Discussion

- The replacement method provides an analysis of logical possibility
- This analysis can also be applied to making the idea of logical consequence more precise
- This was another one of Alfred Tarski's innovations
- So, let's learn how to use the replacement method to test for **logical consequence**

# Introducing FO Consequence

## Logical Consequence

C is a **logical consequence** of  $P_1, \dots, P_n$  iff it is **impossible** for  $P_1, \dots, P_n$  to be true while C is false

- *Impossible* means **logically** impossible
- A logical possibility can be analyzed as pair consisting of a circumstance (state of the world) and a reinterpretation of the nonlogical symbols

## FO Consequence

C is a **FO Consequence** of  $P_1, \dots, P_n$  iff in every circumstance and under every reinterpretation of the non-logical symbols, if  $P_1, \dots, P_n$  come out true, C does too

# FO Consequence An Example

## ARGUMENT 1

$\forall x (\text{Small}(x) \rightarrow \text{Cube}(x))$	
$\text{Small}(a)$	
$\text{Cube}(a)$	

- Let's see if we can find a circumstance and reinterpretation of Argument 1 that makes the premises true and the conclusion false

## ARGUMENT 1'

$\forall x (\text{Nar}(x) \rightarrow \text{Wiv}(x))$	
$\text{Nar}(n)$	
$\text{Wiv}(n)$	

- *All nars are wivs, b is a nar, so n is a wiv*
- This still sounds valid, whatever nars, wivs and  $n$  are

- So,  $\text{Cube}(a)$  is a FO Consequence of the premises

# FO Consequence A Different Example

## ARGUMENT 2

$\text{Cube}(a)$	
$\text{Dodec}(b)$	
$\neg(a = b)$	

- Let's see if we can find a circumstance and reinterpretation of Argument 1 that makes the premises true and the conclusion false
- Let *Rah* mean *is a reporter*, *Bru* mean *is a super-hero*,  $n$  mean *Clark Kent* and  $m$  mean *Superman*
- Now consider the fictional world of the superman comics:

## ARGUMENT 2'

$\text{Rah}(n)$	
$\text{Bru}(m)$	
$\neg(n = m)$	

- So,  $\neg(a = b)$  is **not** a FO Consequence of the premises

- $\text{Rah}(n)$  is true
- $\text{Bru}(m)$  is true
- But  $\neg(n = m)$  is false

# FO Consequence The Replacement Method

## The Replacement Method (FO Consequence)

The following method can be used to check whether or not C is a FO Consequence of  $P_1, \dots, P_n$ :

- 1 Systematically replace all of the non-logical symbols with non-sense symbols
- 2 Try to describe a circumstance, along with interpretations of the predicates in which  $P_1, \dots, P_n$  are true and C false.
  - If there is no such circumstance and interpretation, C is a FO Consequence of  $P_1, \dots, P_n$
  - If there one, it's called a *counterexample* and C is **not** a FO Consequence of  $P_1, \dots, P_n$

# In Class Exercise

## Exercise 10.10

Let's use **FO Con** in Fitch to check our answers

# FO Equivalence

## One Last Thing

### First-Order Equivalence (FO Equivalence)

A and B are **FO equivalent** iff B is a FO consequence of A and A is a FO consequence of B

- So, there's nothing more to FO equivalence than to FO consequence
- To show FO consequence you just use the replacement method to show that A and B are FO consequences of each other