The Logic of Conditionals
Informal & Formal Proofs

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Announcements
For 10.06.11

1. HW6 (practice midterm, 7.12, 7.13, 8.17) is due in class on Thursday 10.13
   - Answers will be posted on Bb Wednesday (10.11)
2. The midterm is on Thursday 10.13
   - You will have until Saturday 10.15 for take-home portion
3. Practice midterm will be reviewed in section
   - Weds 10.12: 1:25-2:15 (Uris 307)
   - Weds 10.13: 8-9pm (Location TBA)
4. HW1-4 will be returned after class

Outline

Material Conditional
Modus Ponens

Truth Table for →

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

Modus Ponens

If you have established P → Q and P, then you can infer Q

- This rule is also known as conditional elimination

- Why, again, is modus ponens correct?
  - If P → Q is T and P is T, then Q **must** be T
  - So when you have P → Q and P, you have Q!

Material Conditional

Modus Ponens at Work

A Simple Application of Modus Ponens

Suppose you are told that if a is a cube, then it is small, and that a is indeed a cube. Then it follows by modus ponens that a is small. Symbolically:

Cube(a) → Small(a) and Cube(a), therefore Small(a).

Modus Ponens Again

Suppose you are told that if a is either a cube or a tetrahedron, then a is in the same row as b, and that a is a cube. Then it follows that a is a cube or a tetrahedron. So by modus ponens, it follows that a is in the same row as b. Symbolically:

We are given that (Cube(a) ∨ Tet(a)) → SameRow(a, b) and Cube(a). By the second claim: Cube(a) ∨ Tet(a) follows. Then by modus ponens it follows that SameRow(a, b).

Conditional Proof

The Method

The Method of Conditional Proof

To prove P → Q, temporarily assume P. If you can show Q with this additional assumption, you can infer P → Q

Truth Table for →

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

- The only way for P → Q to be F is for P to be T and Q be F
- So, if you can show that when P is T Q is also T, you've shown that P → Q is not F; but then it must be T!

Conditional Proof

An Example

Let's use conditional proof and modus ponens to give a proof of:

Argument 1

Tet(a) → Tet(b)
Tet(b) → Tet(c)

Tet(a) → Tet(c)

Our goal is a conditional, so we use conditional proof.

Proof: Suppose Tet(a). Then by premise 1 Tet(b) follows by modus ponens. But then we may now again use modus ponens and premise 2 to infer Tet(c). This is the consequent of our goal, so we have successfully completed our conditional proof.

Conditional Proof

Another Example

Let's do **exercise 8.4** on the chalkboard

8.4
The unicorn, if horned, is elusive and dangerous.
If elusive or mythical, the unicorn is rare.
If a mammal, the unicorn is not rare.

The unicorn, if horned, is not a mammal.

Give an informal proof of the validity of this argument, using conditional proof.

The Material Biconditional

Elimination

Truth Table for ↔

P	Q	P ↔ Q
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional Elimination

If you have established either P ↔ Q or Q ↔ P and P, then you can infer Q.

- This rule is also known as biconditional elimination

- Why is this correct?
  - If P ↔ Q is T and P is T, then Q **must** be T
  - Similarly, if P ↔ Q is T and Q is T, then P is T

The Material Biconditional

Elimination

Biconditional Elimination Example

Suppose you are told that a is in the same column as b if and only if a is a tetrahedron, and that a is tetrahedron. Then by biconditional elimination, it follows that a is in the same column as b. Symbolically:

SameCol(a, b) ↔ Tet(a) and Tet(a), so SameCol(a, b).

Proving Biconditionals

Conditional Proof Twice Over

How to Prove a Biconditional

To prove P ↔ Q, first, use conditional proof to prove P → Q. Then use conditional proof again to prove Q → P. Showing these two conditionals suffices to prove the biconditional.

- How do you prove a biconditional like P ↔ Q?
- We know that P → Q is equivalent to (P → Q) ∧ (Q → P)
- But we know how to prove (P → Q) ∧ (Q → P):
  - Use conditional proof to show P → Q
  - Then use conditional proof to show Q → P

Proving A Biconditional

An Example

Let's give an informal proof of this argument:

| Cube(a) ↔ Cube(b)
| Cube(b) ↔ Cube(c)
|---
| Cube(a) ↔ Cube(c)

Our goal is a biconditional, so we do two conditional proofs.

Proof:

1. First we'll show Cube(a) → Cube(c) by conditional proof. Suppose Cube(a). Then from premise 1 Cube(b) follows by biconditional elimination. From this and premise 2 it follows by biconditional elimination again that Cube(c). So, Cube(a) → Cube(c)
2. Now we'll show Cube(c) → Cube(a) by conditional proof. Suppose Cube(c). Then from premise 2 Cube(b) follows by biconditional elimination. From this and premise 1 it follows by biconditional elimination again that Cube(a). So, Cube(c) → Cube(a).

By these two conditional proofs, it follows that Cube(a) ↔ Cube(c)

Proving a Biconditional

In Class Exercise

Exercise **8.5**: Construct an informal proof of the argument. Here's the argument translated into FOL.

$$(\mathsf{Horned}(u) \to (\mathsf{Elusive}(u) \land \mathsf{Magical}(u))) \land (\neg\mathsf{Horned}(u) \to (\neg\mathsf{Elusive}(u) \land \neg\mathsf{Magical}(u)))$$

$$\neg\mathsf{Horned}(u) \to \neg\mathsf{Mythical}(u)$$

$$\mathsf{Horned}(u) \leftrightarrow (\mathsf{Magical}(u) \lor \mathsf{Mythical}(u))$$

Conditionals

Additional Steps

Some equivalences that are useful for informal proofs w/conditionals:

Important Equivalences

$P \to Q$	$\Longleftrightarrow$	$\neg Q \to \neg P$
$P \to Q$	$\Longleftrightarrow$	$\neg P \lor Q$
$\neg(P \to Q)$	$\Longleftrightarrow$	$P \land \neg Q$
$P \leftrightarrow Q$	$\Longleftrightarrow$	$(P \to Q) \land (Q \to P)$
$P \leftrightarrow Q$	$\Longleftrightarrow$	$(P \land Q) \lor (\neg P \land \neg Q)$

Conditional Elimination

Formalizing Modus Ponens

Modus Ponens

If you have established $P \to Q$ and P, then you can infer Q

- A simple example:

1	$\mathsf{Tet}(a) \to \mathsf{Tet}(b)$	
2	$\mathsf{Tet}(a)$	
3	$\mathsf{Tet}(b)$	$\to$ **Elim**: 1, 2

$\to$ Elim

$$P \to Q$$
$$\vdots$$
$$P$$
$$\vdots$$
$$\triangleright\ Q$$

- $\to$ **Elim** is the formal counterpart to our informal rule called **modus ponens**

An Example

Using $\to$ **Elim**

Let's construct a formal proof for this argument:

8.31

$$(\neg\mathsf{Mythical}(c) \to \mathsf{Mammal}(c)) \land (\mathsf{Mythical}(c) \to \neg\mathsf{Mortal}(c))$$

$$(\neg\mathsf{Mortal}(c) \lor \mathsf{Mammal}(c)) \to \mathsf{Horned}(c)$$

$$\mathsf{Horned}(c) \to \mathsf{Magical}(c)$$

$$\neg\mathsf{Mythical}(c) \lor \mathsf{Mythical}(c)$$

$$\mathsf{Magical}(c)$$

Conditional Introduction

Formalizing Conditional Proof

Conditional Proof

To prove $P \to Q$, temporarily assume P. If you can show Q with this additional assumption, you can infer $P \to Q$ w/o this assumption

$\to$ Intro

| P
| ⋮
| Q
▷ P → Q

- A simple example:

1	b = a	
2	Tet(a)	
3	Tet(b)	= **Elim**: 1, 2
4	Tet(a) → Tet(b)	→ **Intro**: 2-3

Conditional Rules

Another Example with $\to$ **Elim** & $\to$ **Intro**

Let's do exercise **8.32**. This involves a formal version of the informal proof we did for exercise **8.4**. We will use the informal proof to guide us.

$\leftrightarrow$ **Elim**

Formalizing Biconditional Elimination

Biconditional Elimination

If you have established either $P \leftrightarrow Q$ or $Q \leftrightarrow P$, and P, then you can infer Q

$\leftrightarrow$ Elim

| P ↔ Q (*or* Q ↔ P)
| ⋮
| P
| ⋮
▷ Q

- A simple example:

1	Tet(a) ↔ Tet(b)	
2	Tet(a)	
3	Tet(b)	↔ **Elim**: 1, 2

- $\leftrightarrow$ **Elim** is the formal counterpart to our informal rule called **biconditional elimination**

$\leftrightarrow$ **Intro**

Formalizing Biconditional Proof

Biconditional Proof

To prove $P \leftrightarrow Q$ use conditional proof to show $P \to Q$. Then use conditional proof again to show $Q \to P$.

- One subproof amounts to showing $P \to Q$
- The other amounts to showing $Q \to P$

$\leftrightarrow$ Intro

| | P
| | ⋮
| | Q
| | Q
| | ⋮
| | P
▷ P ↔ Q

$\leftrightarrow$ Elim & $\leftrightarrow$ Intro

A Simple Example

Let's do a proof in Fitch for a simple example that uses both $\leftrightarrow$ **Intro** and $\leftrightarrow$ **Elim**:

8.25 *Transitivity of the Biconditional*

$$\begin{array}{|l} A \leftrightarrow B \\ B \leftrightarrow C \\ \hline A \leftrightarrow C \end{array}$$

Informal Proofs Formal Proofs

$\leftrightarrow$ Elim & $\leftrightarrow$ Intro

In Class Exercise

You constructed an informal proof for this argument, now turn this into a formal proof:

8.33

$$\begin{array}{|l} (\mathsf{Horned(c)} \rightarrow (\mathsf{Elusive(c)} \land \mathsf{Magical(c)})) \\ \quad \land (\neg \mathsf{Horned(c)} \rightarrow (\neg \mathsf{Elusive(c)} \land \neg \mathsf{Magical(c)})) \\ \neg \mathsf{Horned(c)} \rightarrow \neg \mathsf{Mythical(c)} \\ \hline \mathsf{Horned(c)} \leftrightarrow (\mathsf{Magical(c)} \lor \mathsf{Mythical(c)}) \end{array}$$

Hint: You should do two subproofs and then apply $\leftrightarrow$ **Intro** to get the conclusion

1. In the first subproof, assume Horned(c), show Magical(c) $\lor$ Mythical(c)
2. In the second one, assume Magical(c) $\lor$ Mythical(c), show Horned(c). It may be easier to show Horned(c) using indirect proof (assume $\neg$Horned(c) and derive $\bot$)