Conditionals

The Basics

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Announcements

For 10.04

1. The midterm is a week from this Thursday!
   - HW6 will be 3 exercises and a practice midterm
   - The practice midterm has the exact format of the real midterm
   - The practice midterm will be posted to Bb today
2. HW1-4 will be returned on Thursday!
   - Do not miss this!

Outline

Introduction

Beyond the Booleans

- We've learned how to treat the Boolean connectives (*and*, *or*, *not*) in FOL
  - Using the connectives $\wedge, \vee, \neg$ we've learned:
    1. How to translate sentence of English with *and*, *or* and *not* into FOL
    2. What these sentences mean, using truth tables and proofs for $\wedge, \vee, \neg$
- But the Booleans are just the tip of the iceberg!
- There are many other logically important constructions in natural language that we need to learn to treat in FOL

Introduction
Other Connectives

- Consider these sentences:
  (1) Herb is wearing fake alligator boots if he is going to the market
  (2) Josh is surfing only if he is in California
  (3) Maria is doing math if and only if she is in NYC
  (4) Jerry will go to the opera unless it is Verde
  (5) Susan quit because she found a better career
- These all contain a logical connective you haven't studied yet

Introduction
Today's Topics

- Today we are going to learn how to treat 4 of those 5 connectives (and a few others) in FOL
- The 4 we will cover:
  1. *if*
  2. *only if*
  3. *if and only if*
  4. *unless*
- These can be treated in FOLwith only two new connectives: $\rightarrow, \leftrightarrow$
- Why are we ignoring *because*?

Introduction
Truth-Functionality

- *Because* is not truth functional
- What does *truth functional* mean?
  - Suppose we have some binary sentential connective $\star$
  - To say that $\star$ is truth functional is just to say that the truth value of $\mathsf{A} \star \mathsf{B}$ is completely determined by the truth values of A and B
  - $\wedge$ is truth functional:
    - $\mathsf{A} \wedge \mathsf{B}$ is true if and only if A is true and B is true, and false otherwise
- To see exactly why *because* is not truth functional read pp.177 of *LPL*
- Non-truth functional connectives are not defective
  - They are just very hard to analyze and require more sophisticated tools than those taught in this class

Introduction
The Plan for Today

- Today we'll add two new connectives to FOL: $\rightarrow$ and $\leftrightarrow$
- We'll learn what then mean in terms of truth tables and game rules
- Then we'll learn how do use them to translate sentences containing *if*, *only if*, *if and only if*, *unless* and a few other English connectives

The Basics
Grammar and Terminology

- The → symbol is used to combine two sentences P and Q to form a new sentence:

  (6) P → Q
- A sentence like (6) is called a **material conditional**
- P is called the **antecedent**
- Q is called the **consequent**
- We will discuss the English counterparts of → after we learn what it means
- For now, we'll read P → Q as *If P then Q*

The Basics
The Meaning of →

Truth Table for →

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

- P → Q is T when the truth of P somehow guarantees that Q will be T
- P → Q is F when P is T and Q is F
- Otherwise, P → Q is T

Game Rule for →

When playing a game Tarski's World treats P → Q as an abbreviation for ¬P ∨ Q. This means:

1. If you commit to the truth of P → Q, you commit to the falsity of P or the truth of Q
2. If you commit to the falsity of P → Q, you commit to the truth of P and the falsity of Q

The Basics
More on the Meaning of →

Truth Table for →

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

- This table for → makes P → Q tautologically equivalent to ¬P ∨ Q
- Let's see this in detail using Boole to construct a joint truth table
- This table also says that P → Q is F just in case P is T and Q is F
  - So ¬(P → Q) and P ∧ ¬Q should be equivalent
  - Again, let's see what Boole says!
- Now let's look at some sentences in Tarski's World to see if we understand when material conditionals are true

The Basics
Grammar and Terminology

- The ↔ symbol is used to combine two sentences P and Q to form a new sentence:

  (7) P ↔ Q
- A sentence like (7) is called a **material biconditional**
- The most common way to read (7) is:

  P if and only if Q
- It is common to abbreviate *if and only if* as *iff*
- Now, what does ↔ mean?

The Basics
The Meaning of $\leftrightarrow$

Truth Table for $\leftrightarrow$

P	Q	P $\leftrightarrow$ Q
T	T	T
T	F	F
F	T	F
F	F	T

- P $\leftrightarrow$ Q is T when P and Q have the same truth value
- P $\leftrightarrow$ Q is F when P and Q have different truth values

Game Rule for $\leftrightarrow$

When playing the game, Tarski's World treats P $\leftrightarrow$ Q as an abbreviation for (P $\to$ Q) $\land$ (Q $\to$ P). This means that committing to the truth of P $\leftrightarrow$ Q commits you to the truth of... something complex. But it is equivalent to committing yourself to P and Q having the same truth value.

The Basics
More on the Meaning of $\leftrightarrow$

Truth Table for $\leftrightarrow$

P	Q	P $\leftrightarrow$ Q
T	T	T
T	F	F
F	T	F
F	F	T

- This table for $\leftrightarrow$ makes P $\leftrightarrow$ Q tautologically equivalent to (P $\to$ Q) $\land$ (Q $\to$ P)
- We'll show it w/Boole
- This table also says that P $\leftrightarrow$ Q is T just in case P and Q have the same truth values
  - So P $\leftrightarrow$ Q and (P $\land$ Q) $\lor$ ($\neg$P $\land$ $\neg$Q) should be equivalent; let's turn to Boole again!
- Let's look at some sentences in Tarski's World to see if we understand when material biconditionals are true

Conditionals
In-Class Exercise

A world and some sentences with be displayed in Tarski's World. It is your job to correctly determine the truth value of each sentence in the world displayed.

Translation with $\to$
Reading P $\to$ Q

- Perhaps the most natural way to read:

  (8) P $\to$ Q

  is:

  (8′) *If P then Q*
- So

  (9) Tet(a) $\to$ Smaller(a, b)

  Could be read as:

  (9′) *If **a** is a tetrahedron, then **a** is smaller than **b***
- Remember that $\to$ can be used to represent English connectives other than *if. . . then*

Translation with →

Varieties of Conditionals

- We've already seen that:
  (8′) *If P then Q*
  Gets represented in FOL as $P \to Q$
- But so do all of the following:
  (10) *Q if P*
  (11) *P only if Q*
  (12) *Q provided P*
- Also, *P unless Q* translates as $\neg Q \to P$
- Let's look at some concrete examples of these sentences and learn how to find their correct translations

Translation with →

P if Q

How should we translate:

(13) Homer drinks if Bart fails in school

First, recognize the form:

- (13) has the form *P if Q*
- *P if Q* translates as $Q \to P$

Second, we translate the parts, *P* and *Q*:

- ***P***: *Homer drinks*, so P : Drinks(homer)
- ***Q***: *Bart fails in school*, so Q : Fails(bart)

Last, we plug our translations of *P* and *Q* into the form:

	Q	→	P
✓	Fails(bart)	→	Drinks(homer)

Translation with →

P only if Q

Translate:

(14) Homer drinks only if Lenny pays

First, the form:

- (14) has the form *P only if Q*
- *P only if Q* translates as $P \to Q$

Second, we translate the parts:

- *P*: *Homer drinks*, so P : Drinks(homer)
- *Q*: *Lenny pays*, so Q : Pays(lenny)

Last, we plug the translation of the parts into the form:

	P	→	Q
✓	Drinks(homer)	→	Pays(lenny)

Translation with →

P provided Q

Translate:

(15) Homer drinks provided Moe is alive

First, the form:

- (15) has the form *P provided Q*
- *P provided Q* translates as $Q \to P$

Second, translate the parts:

- *P*: *Homer drinks*, so P : Drinks(homer)
- *Q*: *Moe is alive*, so Q : Alive(moe)

Last, plug the parts into the form:

	Q	→	P
✓	Alive(moe)	→	Drinks(homer)

Translation with $\to$

P unless Q

Translate:

(16) Garfield is asleep unless he is eating lasagna

First, the form:

- (16) has the form *P unless Q*
- *P unless Q* translates as $\neg Q \to P$

Second, translate the parts:

- *P*: *Garfield is sleeping*, so P : Asleep(garfield)
- *Q*: *He is eating lasagna*, so Q : EatingLas(garfield)

Last, plug the parts into the form:

$$\neg Q \to P$$

✓ $\neg\text{EatingLas(garfield)} \to \text{Asleep(garfield)}$

Translation with $\to$

Summary

English Form	FOL Translation
If P then Q	$P \to Q$
P only if Q	$P \to Q$
P if Q	$Q \to P$
P provided Q	$Q \to P$
P unless Q	$\neg Q \to P$

Translation Recipe (For Any Sentence, Not Just Conditionals)

1. Recognize the form

   Example ***a*** *is a cube unless it is small*
   The Form: *P unless Q* $\leadsto$ $\neg Q \to P$

2. Translate the parts
3. Plug the parts into the form

Translation

Conditionals Compounded

Translate:

(17) ***c*** *is to the right of* ***d*** *only if* ***d*** *is either a cube or small*

First, recognize the form and how it's translated:

- *P only if Q* $\leadsto$ $P \to Q$

Second, translate the parts:

- *P*: ***c*** *is to the right of* ***d***, so P : RightOf(c, d)
- *Q*: ***d*** *is either a cube or small*
  So, Q : Cube(d) $\lor$ Small(d)

Plug these translations into the form:

$$P \to Q$$

✓ $\text{RightOf(c, d)} \to (\text{Cube(d)} \lor \text{Small(d)})$

Translation

Conditionals Compounded

(18) *If* ***e*** *is a cube, then it's to the left of* ***a*** *unless* ***a*** *is small*

Recognize the form:

- *If P then Q* $\leadsto$ $P \to Q$

Now, the parts:

- *P*: ***e*** *is a cube*, so P : Cube(e)
- *Q*: ***e*** *is to the left of* ***a*** *unless* ***a*** *is small*

  The form:
  - *R unless S* $\leadsto$ $\neg S \to R$

  Translate the parts:
  - *R*: ***e*** *is to the left of* ***a***, so R : LeftOf(e, a)
  - *S*: ***a*** *is small*, so S : Small(a)

  Plug S and R into form to get Q: $\neg\text{Small(a)} \to \text{LeftOf(e, a)}$

Now plug P and Q into $P \to Q$: $\text{Cube(e)} \to (\neg\text{Small(a)} \to \text{LeftOf(e, a)})$

Compound Conditionals
Summary

Translation Recipe (For Any Sentence, Not Just Conditionals)

1. Recognize the form

   Example *a is a cube unless it is small*

   The Form: *P unless Q* $\rightsquigarrow$ $\neg Q \to P$
2. Translate the parts
   - If the parts themselves contain connectives, apply the recipe to the parts to translate them
3. Plug the parts into the form

Let's look at some more examples in Tarski's World

Translation
In Class Exercise

Translate:

(19) *If neither Max nor Claire fed Folly at 2:00, then she was hungry*

$\neg(\mathsf{Fed}(\mathsf{max}, \mathsf{folly}, 2{:}00) \lor \mathsf{Fed}(\mathsf{claire}, \mathsf{folly}, 2{:}00)) \to \mathsf{Hungry}(\mathsf{folly}, 2{:}00)$

(20) *If **b** is a dodecahedron, then if it isn't in front of **d** then it isn't in back of **d** either*

$\mathsf{Dodec}(\mathsf{b}) \to (\neg\mathsf{FrontOf}(\mathsf{b}, \mathsf{d}) \to \neg\mathsf{BackOf}(\mathsf{b}, \mathsf{d}))$

(21) *At least one of **a**, **c** and **e** is a cube*

$\mathsf{Cube}(\mathsf{a}) \lor \mathsf{Cube}(\mathsf{c}) \lor \mathsf{Cube}(\mathsf{e})$

(22) ***a** is a tetrahedron unless neither **c** nor **b** are small*

$\neg\neg(\mathsf{Small}(\mathsf{c}) \lor \mathsf{Small}(\mathsf{b})) \to \mathsf{Tet}(\mathsf{a})$

(23) *If **c** and **d** are both cubes, then one is to the right of the other*

$(\mathsf{Cube}(\mathsf{c}) \land \mathsf{Cube}(\mathsf{d})) \to (\mathsf{RightOf}(\mathsf{c}, \mathsf{d}) \lor \mathsf{RightOf}(\mathsf{d}, \mathsf{c}))$

Translation
It's Pretty Simple

- *if and only if* translates as $\leftrightarrow$
- The 'mathematical' use of *just in case* translates as $\leftrightarrow$
- For Example:

  ***a** is a cube if and only if it is small*

  Translates as:

  $\mathsf{Cube}(\mathsf{a}) \leftrightarrow \mathsf{Small}(\mathsf{a})$

  And:

  ***e** is large just in case **d** is small*

  Translates as:

  $\mathsf{Large}(\mathsf{e}) \leftrightarrow \mathsf{Small}(\mathsf{d})$

Conclusion
For Today

- We reviewed the meanings of $\to$ and $\leftrightarrow$:
  - $\mathsf{P} \to \mathsf{Q}$ is F if P is *t* and *Q* F. Otherwise, it is T.
  - $\mathsf{P} \leftrightarrow \mathsf{Q}$ is T just in case P and Q have the same truth value. Otherwise, it is F
- We learned how to translate several kinds of English conditionals using $\to$ and $\leftrightarrow$
- We learned a recipe for translating sentences containing connectives
- We saw that the recipe really comes into its own when the connective connects compound sentences