

Formal Proofs and Boolean Logic II

Extending \mathcal{F} with Rules for \neg

William Starr

09.29.11

Review

Proof by Contradiction

- Last class: formal proofs for \vee and \wedge
 - What about \neg ?
- That's the topic of Today's class
- Our \neg **Intro** rule will allow us to prove negated claims
 - Just like proof by contradiction!
- So let's review that informal method

Outline

- 1 Review
- 2 Formal Rules for \neg
- 3 Using Subproofs
- 4 Proof Strategies
- 5 Conclusion

Proof by Contradiction

Proving a Negated Claim

Proof by Contradiction (Official Version)

- 1 To prove that P is false, show that a contradiction \perp follows from P
- 2 To prove that P is true, show that a contradiction \perp follows from $\neg P$

Proving a Negated Claim

To prove $\neg P$, assume P and prove a contradiction \perp

- All contradictions are **impossible**, thus **false**
- If you can show that P leads to a contradiction, then P must be false
- But if P is false, then $\neg P$ must be true

Review

What is a Contradiction Again?

Contradiction

- A **contradiction** is any sentence that cannot possibly be true, or any group of sentences that cannot all be true simultaneously
- The symbol \perp is often used as a short-hand way of saying that a contradiction has been obtained

• Examples:

- 1 $\neg\text{Cube}(a) \wedge \text{Cube}(a)$
- 2 $a = b, b = c, a \neq c$
- 3 $\text{Cube}(a) \wedge \text{Tet}(a)$

Proof by Contradiction

A Simple Example

Claim: This argument is valid

$$\begin{array}{l} \neg\text{SameShape}(a, b) \\ b = c \\ \hline \neg a = c \end{array}$$

Proof: We want to show $\neg a = c$ from the premises, so we will use a **proof by contradiction**

- 1 Suppose $a = c$
- 2 Then, from **premise one** $\neg\text{SameShape}(c, b)$ follows by Indiscernibility of Identicals
- 3 But by **premise two**, we know $\text{SameShape}(c, b)$. This is a contradiction, \perp !
- 4 So our **supposition** must have been **false**; that is, $\neg a = c$ must be **true** given the premises

Formal Rules for \neg

Where We Are Going

- The basic idea behind \neg **Intro** is familiar from our informal method of proof by contradiction
 - You can use \neg **Intro** to infer $\neg P$ when you have proven that a contradiction \perp follows from P
- What exactly counts as proving a contradiction (\perp)?
- If we had a \perp **Intro** rule, when should we apply it?

Two Kinds of Contradictions

Boolean vs. Analytic

Boolean Contradictions

- E.g. $\text{Cube}(a), \neg\text{Cube}(a)$ or $\text{Tet}(a) \wedge \neg\text{Tet}(a)$
- Can't be true because of what the **Booleans mean**

VS

Analytic Contradictions

- E.g. $\text{Large}(a), \text{Small}(a)$ or $\text{FrontOf}(a, b), \text{BackOf}(a, b)$
- Can't be true because of what the **predicates mean**

Contradictions

\perp Intro

\perp Intro	
P	
⋮	
$\neg P$	
▷	\perp

- So, you've proven P and $\neg P$?
 - You can introduce \perp
- **Question:** does this rule detect **Analytic** contradictions? (Like FrontOf(a, b), BackOf(a, b))
- **Answer:** NO!!

- **Question:** How would you infer \perp on the basis of FrontOf(a, b), BackOf(a, b)?
- **Answer:** In Fitch, you can do it with **Ana Con**

Boolean v. Analytic Contradictions

Within \mathcal{F}

Boolean \perp in \mathcal{F}

1	Cube(a)	
2	\neg Cube(a)	
3	\perp	\perp Intro: 1, 2

- We have P and $\neg P$
- So \perp **Intro** allows us to introduce \perp

Analytic \perp in Fitch

1	Cube(a)	
2	Tet(a)	
3	\perp	Ana Con: 1, 2

- Here we do **not** have P and $\neg P$
- So \perp **Intro** does not give us \perp
- But **Ana Con** does

\perp Elim

What Should \perp Elim Be?

- Remember, all rules come in pairs
- We've stated \perp **Intro**, but we haven't said anything about \perp **Elim**
- What **should** we be able to infer from a contradiction?
- Let's think about it for a minute

Valid Arguments

What If The Premises are Inconsistent?

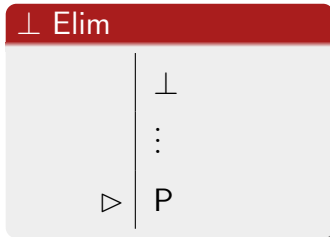
Logical Consequence, Validity

C is a logical consequence of P_1, \dots, P_n if and only if it is **impossible** for P_1, \dots, P_n to be true while C false

- What follows from a contradiction?
 - Anything!
- Why?
 - It's impossible for it to be true
- So, it is impossible for it to be true while **any** conclusion is false!

Contradictions

\perp Elim



- From a contradiction \perp , any **conclusion** follows!
- Why again?

- An inference step is valid just in case it **cannot** lead you from a **true** premise to a false conclusion
- Since the premise \perp in this inference can never be true, the inference can never lead one from a true premise to a false conclusion

Contradictions

Wait, What were We Doing?

- So, two more rules in \mathcal{F} : \perp **Intro**, \perp **Elim**
- Cool, but why did go on this tangent about \perp ?
- Because introducing \perp was essential for \neg **Intro**
 - \neg **Intro** is proof by contradiction, so we needed to know exactly when we could write \perp
- So now we are in a position to see \neg **Intro**

\neg Intro

From Informal to Formal Proof

Proving a Negative Claim

- To prove $\neg P$, assume P and prove a contradiction using this assumption
- This is an example of **Proof by Contradiction**

Example Informal Proof

From $a = b$ and $b \neq c$ we will prove $a \neq c$. We use proof by contradiction. **Proof:** Suppose $a = c$. Well, $b = c$ follows from this assumption and premise one by Ind. of Id.'s. But, this contradicts premise two, \perp . So our assumption was wrong, in which case $a \neq c$.

\neg Intro

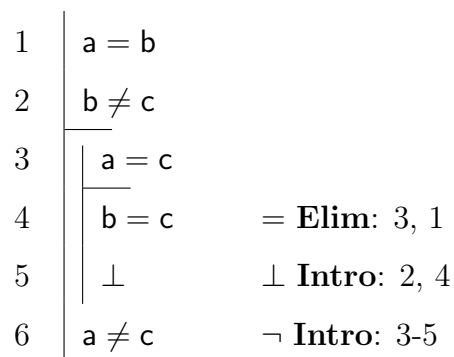


To **prove** $\neg P$:

- 1 Assume P
- 2 Derive \perp (using \perp **Intro**)
- 3 Conclude $\neg P$ (Discharging assumption of P)

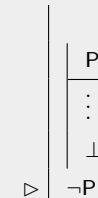
\neg Intro

An Example

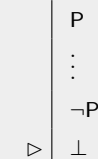


✓ **Goal:** $a \neq c$

\neg Intro



\perp Intro



Some More Examples

- Let's do a formal proof for 6.25:

$$\frac{\neg A \wedge \neg B}{\neg(A \vee B)}$$

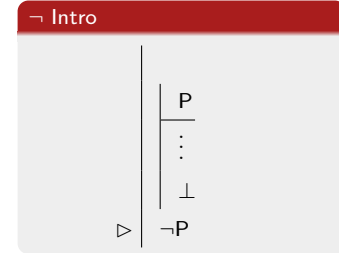
- Let's also finish the proof from slide 26 of 02.19
 - This will use \perp **Elim**

\neg Intro

Another Example

ARGUMENT 1: Analytic \perp Revisited

1	$\neg\text{SameShape}(a, b)$	
2	$b = c$	
3	$a = c$	
4	$\neg\text{SameShape}(c, b)$	= Elim : 1,3
5	\perp	Ana Con : 2, 4
6	$a \neq c$	\neg Intro : 3-5



Goal: $a \neq c$

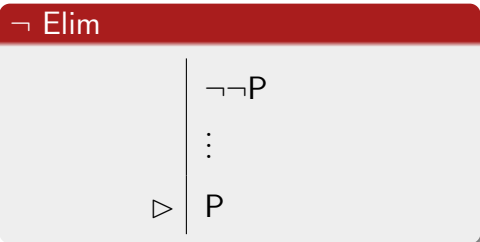
Informal Proof

We want to show $a \neq c$, so we use proof by contradiction. **Proof**: Suppose $a = c$. From premise one it follows that $\neg\text{SameShape}(c, b)$, by Ind. of Id. But this contradicts premise two which requires that c is b . So our assumption ($a = c$) was wrong, hence $a \neq c$ follows.

- There is no rule in \mathcal{F} which justifies line 5
- But this is what we need to prove $a \neq c$!
- So, this proof can't be finished in \mathcal{F}
- We *can* finish it in **Fitch**!

Negation

\neg Elim



- If $\neg\neg P$ is true, so is P
- Obvious and useless? No!

Simple Example

1	$\neg\neg\text{Cube}(a)$	
2	$\text{Cube}(a)$	\neg Elim : 1

- Its use: prove P by contradiction
- Use \neg **Intro** to prove $\neg\neg P$, then apply \neg **Elim**

\neg Elim

An Example

ARGUMENT 2

$\text{Tet}(e) \vee \text{Cube}(a)$
$\neg\text{Tet}(e)$
$\text{Cube}(a)$

Informal Proof of Argument 2

We will use a proof by contradiction. Suppose $\neg\text{Cube}(a)$. This pretty clearly contradicts the premises. To be sure, we'll take it in cases. Suppose $\text{Tet}(e)$. Then the contradiction is clear. Suppose $\neg\text{Cube}(a)$. Then we also have a contradiction. So our assumption must have been wrong. Hence, $\text{Cube}(a)$ must be true given the premises.

Let's make this into a formal proof in Fitch

Subproofs

The Big Picture

- Subproofs correspond to elements of informal proofs:
 - The **cases** of a **proof by cases**
 - The **temporary assumption** in a **proof by contradiction**
- Just like cases and temporary assumptions, there are certain **important restrictions** on subproofs

Cases

The Constraints

ARGUMENT 3

$$\frac{(\text{Cube}(c) \wedge \text{Small}(c)) \vee (\text{Tet}(c) \wedge \text{Small}(c))}{\text{Small}(c) \wedge \text{Cube}(c) \wedge \text{Tet}(c)}$$

Pseudo-Proof of Argument 3

We will use a proof by cases based on premise one. **Case 1:** Suppose $(\text{Cube}(c) \wedge \text{Small}(c))$. Then $\text{Small}(c)$ follows. **Case 2:** Suppose $\text{Tet}(c) \wedge \text{Small}(c)$. Then $\text{Small}(c)$ follows. So, $\text{Small}(c)$ follows in either case. But in case 1 we had $\text{Cube}(c)$ and in case 2 we had $\text{Tet}(c)$, hence our conclusion follows: $\text{Small}(c) \wedge \text{Cube}(c) \wedge \text{Tet}(c)$.

- Why **pseudo-proof**?
 - Argument 3 is **not valid**
 - This ‘proof’ leads us from a **possible premise** to an **impossible conclusion**
 - That’s exactly what proofs **aren’t supposed to do**

Cases

The Constraint

Pseudo-Proof of Argument 3

We will use a proof by cases based on premise one. **Case 1:** Suppose $(\text{Cube}(c) \wedge \text{Small}(c))$. Then $\text{Small}(c)$ follows. **Case 2:** Suppose $\text{Tet}(c) \wedge \text{Small}(c)$. Then $\text{Small}(c)$ follows. So, $\text{Small}(c)$ follows in either case. **But in case 1 we had $\text{Cube}(c)$ and in case 2 we had $\text{Tet}(c)$, hence our conclusion follows: $\text{Small}(c) \wedge \text{Cube}(c) \wedge \text{Tet}(c)$.**

- Where exactly does this proof go wrong?
 - We picked a claim out of a case **after it was finished**
 - The assumptions and conclusions of a case are only available **within that case**

The Moral

What happens in a case, stays in a case.

Temporary Assumptions

The Constraints

- In proof by contradiction, like in proof by cases, we make a temporary assumption:
 - We assume P and try to show \perp
 - But P is a **temporary assumption**
 - So anything we infer from it is also **temporary**
- Once we show \perp , we **discharge** the assumption of P
- This temporary assumption of P , and the things we infer from it, corresponds to a subproof
- Once this assumption is **discharged**, we can’t reach back into the subproof

Subproofs

Drawing the Connections

Proof with A Subproof

$$\begin{array}{l} \vdots \\ | \quad A \\ | \quad \vdots \\ | \quad B \\ \vdots \\ C \end{array}$$

- A subproof involves a **temporary assumption**
 - Like proof by contradiction
 - Like proof by cases
- So you can't reiterate lines from the subproof outside of the subproof

Subproofs

Guidelines for Use

Guidelines for Using Subproofs

- 1 Once a subproof has ended, you can **never** cite one of its lines individually for any purpose, although you may cite the subproof as a whole (as in \vee **Elim** & \neg **Intro**)
- 2 In justifying a step of a proof, you may cite any earlier line of the **main proof**, or any subproof that **has not ended**

Let's do exercise 6.17 to solidify these points

Proof Strategies

How to Approach a Formal Proof

- 1 Understand what the sentences are saying
- 2 Decide whether you think the conclusion follows from the premises
- 3 If you don't think so, try to find a counterexample
- 4 If you do think so, try to give an informal proof
- 5 Use this informal proof to guide your formal proof
- 6 If you get stuck try working backwards

Summary

Negation

- We learned two new negation rules: \neg **Intro**, \neg **Elim**
- \neg **Intro** mirrors the proof by contradiction method
- To mimic this method in \mathcal{F} we introduced the \perp symbol and two rules for it: \perp **Intro**, \perp **Elim**
- Proof by contradiction isn't just good for proving negated claims
 - It can also be used to prove positive claims

Summary

Subproofs & Strategy

- Mastering \mathcal{F} involves mastering subproofs
- Just like cases and reductio assumptions, there are constraints on how you can use subproofs
- We learned these constraints and the perils the guard us against
- We also learned how to approach proofs
 - There's strategy to it!
 - **Don't** just try to shuffle symbols!