

# Formal Proofs and Boolean Logic II

Extending  $\mathcal{F}$  with Rules for  $\neg$

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## Review

Proof by Contradiction

- Last class: formal proofs for  $\vee$  and  $\wedge$ 
  - What about  $\neg$ ?
- That's the topic of Today's class
- Our  $\neg$  **Intro** rule will allow us to prove negated claims
  - Just like proof by contradiction!
- So let's review that informal method

## Outline

- 1 Review
- 2 Formal Rules for  $\neg$
- 3 Using Subproofs
- 4 Proof Strategies
- 5 Conclusion

## Proof by Contradiction

Proving a Negated Claim

### Proof by Contradiction (Official Version)

- 1 To prove that  $P$  is false, show that a contradiction  $\perp$  follows from  $P$
- 2 To prove that  $P$  is true, show that a contradiction  $\perp$  follows from  $\neg P$

### Proving a Negated Claim

To prove  $\neg P$ , assume  $P$  and prove a contradiction  $\perp$

- All contradictions are **impossible**, thus **false**
- If you can show that  $P$  leads to a contradiction, then  $P$  must be false
- But if  $P$  is false, then  $\neg P$  must be true

## Review

What is a Contradiction Again?

### Contradiction

- A **contradiction** is any sentence that cannot possibly be true, or any group of sentences that cannot all be true simultaneously
- The symbol  $\perp$  is often used as a short-hand way of saying that a contradiction has been obtained

• Examples:

- 1  $\neg\text{Cube}(a) \wedge \text{Cube}(a)$
- 2  $a = b, b = c, a \neq c$
- 3  $\text{Cube}(a) \wedge \text{Tet}(a)$

## Proof by Contradiction

A Simple Example

**Claim:** This argument is valid

$$\begin{array}{l} \neg\text{SameShape}(a, b) \\ b = c \\ \hline \neg a = c \end{array}$$

**Proof:** We want to show  $\neg a = c$  from the premises, so we will use a **proof by contradiction**

- 1 Suppose  $a = c$
- 2 Then, from **premise one**  $\neg\text{SameShape}(c, b)$  follows by Indiscernibility of Identicals
- 3 But by **premise two**, we know  $\text{SameShape}(c, b)$ . This is a contradiction,  $\perp$ !
- 4 So our **supposition** must have been **false**; that is,  $\neg a = c$  must be **true** given the premises

## Formal Rules for $\neg$

Where We Are Going

- The basic idea behind  $\neg$  **Intro** is familiar from our informal method of proof by contradiction
  - You can use  $\neg$  **Intro** to infer  $\neg P$  when you have proven that a contradiction  $\perp$  follows from  $P$
- What exactly counts as proving a contradiction ( $\perp$ )?
- If we had a  $\perp$  **Intro** rule, when should we apply it?

## Two Kinds of Contradictions

Boolean vs. Analytic

### Boolean Contradictions

- E.g.  $\text{Cube}(a), \neg\text{Cube}(a)$  or  $\text{Tet}(a) \wedge \neg\text{Tet}(a)$
- Can't be true because of what the **Booleans mean**

VS

### Analytic Contradictions

- E.g.  $\text{Large}(a), \text{Small}(a)$  or  $\text{FrontOf}(a, b), \text{BackOf}(a, b)$
- Can't be true because of what the **predicates mean**

## Contradictions

$\perp$  Intro

$\perp$ Intro	
P	
⋮	
$\neg P$	
▷	$\perp$

- So, you've proven P and  $\neg P$ ?
  - You can introduce  $\perp$
- **Question:** does this rule detect **Analytic** contradictions? (Like FrontOf(a, b), BackOf(a, b))
- **Answer:** NO!!

- **Question:** How would you infer  $\perp$  on the basis of FrontOf(a, b), BackOf(a, b)?
- **Answer:** In Fitch, you can do it with **Ana Con**

## Boolean v. Analytic Contradictions

Within  $\mathcal{F}$

### Boolean $\perp$ in $\mathcal{F}$

1	Cube(a)	
2	$\neg$ Cube(a)	
3	$\perp$	$\perp$ Intro: 1, 2

- We have P and  $\neg P$
- So  $\perp$  **Intro** allows us to introduce  $\perp$

### Analytic $\perp$ in Fitch

1	Cube(a)	
2	Tet(a)	
3	$\perp$	<b>Ana Con:</b> 1, 2

- Here we do **not** have P and  $\neg P$
- So  $\perp$  **Intro** does not give us  $\perp$
- But **Ana Con** does

## $\perp$ Elim

What Should  $\perp$  Elim Be?

- Remember, all rules come in pairs
- We've stated  $\perp$  **Intro**, but we haven't said anything about  $\perp$  **Elim**
- What **should** we be able to infer from a contradiction?
- Let's think about it for a minute

## Valid Arguments

What If The Premises are Inconsistent?

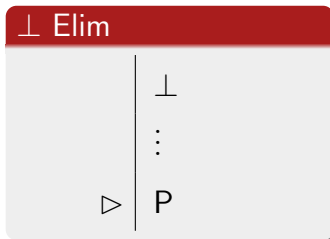
### Logical Consequence, Validity

C is a logical consequence of  $P_1, \dots, P_n$  if and only if it is **impossible** for  $P_1, \dots, P_n$  to be true while C false

- What follows from a contradiction?
  - Anything!
- Why?
  - It's impossible for it to be true
- So, it is impossible for it to be true while **any** conclusion is false!

# Contradictions

$\perp$  Elim



- From a contradiction  $\perp$ , any **conclusion** follows!
- Why again?

- An inference step is valid just in case it **cannot** lead you from a **true** premise to a false conclusion
- Since the premise  $\perp$  in this inference can never be true, the inference can never lead one from a true premise to a false conclusion

# Contradictions

Wait, What were We Doing?

- So, two more rules in  $\mathcal{F}$ :  $\perp$  **Intro**,  $\perp$  **Elim**
- Cool, but why did go on this tangent about  $\perp$ ?
- Because introducing  $\perp$  was essential for  $\neg$  **Intro**
  - $\neg$  **Intro** is proof by contradiction, so we needed to know exactly when we could write  $\perp$
- So now we are in a position to see  $\neg$  **Intro**

# $\neg$ Intro

From Informal to Formal Proof

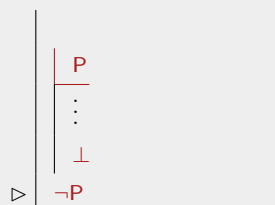
## Proving a Negative Claim

- To prove  $\neg P$ , assume  $P$  and prove a contradiction using this assumption
- This is an example of **Proof by Contradiction**

## Example Informal Proof

From  $a = b$  and  $b \neq c$  we will prove  $a \neq c$ . We use proof by contradiction. **Proof:** Suppose  $a = c$ . Well,  $b = c$  follows from this assumption and premise one by Ind. of Id.'s. But, this contradicts premise two,  $\perp$ . So our assumption was wrong, in which case  $a \neq c$ .

## $\neg$ Intro

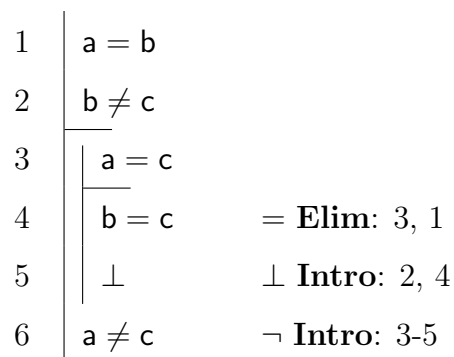


To **prove**  $\neg P$ :

- 1 Assume  $P$
- 2 Derive  $\perp$  (using  $\perp$  **Intro**)
- 3 Conclude  $\neg P$  (Discharging assumption of  $P$ )

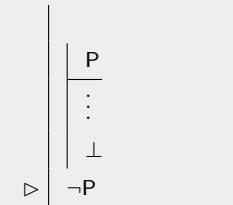
# $\neg$ Intro

An Example

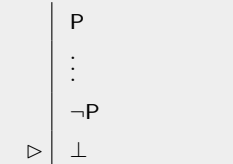


✓ **Goal:**  $a \neq c$

## $\neg$ Intro



## $\perp$ Intro



# Some More Examples

- Let's do a formal proof for 6.25:

$$\frac{\neg A \wedge \neg B}{\neg(A \vee B)}$$

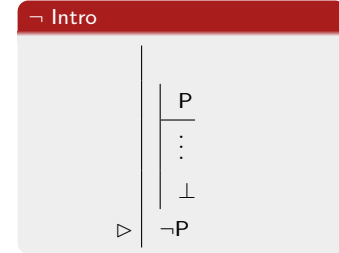
- Let's also finish the proof from slide 26 of 02.19
  - This will use  $\perp$  **Elim**

# $\neg$ Intro

## Another Example

ARGUMENT 1: Analytic  $\perp$  Revisited

1	$\neg\text{SameShape}(a, b)$	
2	$b = c$	
3	$a = c$	
4	$\neg\text{SameShape}(c, b)$	= <b>Elim</b> : 1,3
5	$\perp$	<b>Ana Con</b> : 2, 4
6	$a \neq c$	$\neg$ <b>Intro</b> : 3-5



Goal:  $a \neq c$

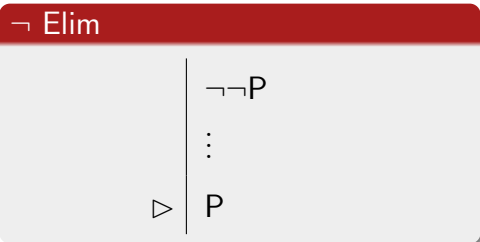
### Informal Proof

We want to show  $a \neq c$ , so we use proof by contradiction. **Proof**: Suppose  $a = c$ . From premise one it follows that  $\neg\text{SameShape}(c, b)$ , by Ind. of Id. But this contradicts premise two which requires that  $c$  is  $b$ . So our assumption ( $a = c$ ) was wrong, hence  $a \neq c$  follows.

- There is no rule in  $\mathcal{F}$  which justifies line 5
- But this is what we need to prove  $a \neq c$ !
- So, this proof can't be finished in  $\mathcal{F}$
- We *can* finish it in **Fitch**!

# Negation

## $\neg$ Elim



- If  $\neg\neg P$  is true, so is  $P$
- Obvious and useless? No!

### Simple Example

1	$\neg\neg\text{Cube}(a)$	
2	$\text{Cube}(a)$	$\neg$ <b>Elim</b> : 1

- Its use: prove  $P$  by contradiction
- Use  $\neg$  **Intro** to prove  $\neg\neg P$ , then apply  $\neg$  **Elim**

# $\neg$ Elim

## An Example

ARGUMENT 2

$\text{Tet}(e) \vee \text{Cube}(a)$
$\neg\text{Tet}(e)$
$\text{Cube}(a)$

### Informal Proof of Argument 2

We will use a proof by contradiction. Suppose  $\neg\text{Cube}(a)$ . This pretty clearly contradicts the premises. To be sure, we'll take it in cases. Suppose  $\text{Tet}(e)$ . Then the contradiction is clear. Suppose  $\neg\text{Cube}(a)$ . Then we also have a contradiction. So our assumption must have been wrong. Hence,  $\text{Cube}(a)$  must be true given the premises.

Let's make this into a formal proof in Fitch

## Subproofs

### The Big Picture

- Subproofs correspond to elements of informal proofs:
  - The **cases** of a **proof by cases**
  - The **temporary assumption** in a **proof by contradiction**
- Just like cases and temporary assumptions, there are certain **important restrictions** on subproofs

## Cases

### The Constraints

ARGUMENT 3

$$\frac{(Cube(c) \wedge Small(c)) \vee (Tet(c) \wedge Small(c))}{Small(c) \wedge Cube(c) \wedge Tet(c)}$$

#### Pseudo-Proof of Argument 3

We will use a proof by cases based on premise one. **Case 1:** Suppose  $(Cube(c) \wedge Small(c))$ . Then  $Small(c)$  follows. **Case 2:** Suppose  $Tet(c) \wedge Small(c)$ . Then  $Small(c)$  follows. So,  $Small(c)$  follows in either case. But in case 1 we had  $Cube(c)$  and in case 2 we had  $Tet(c)$ , hence our conclusion follows:  $Small(c) \wedge Cube(c) \wedge Tet(c)$ .

- Why **pseudo-proof**?
  - Argument 3 is **not valid**
  - This ‘proof’ leads us from a **possible premise** to an **impossible conclusion**
  - That’s exactly what proofs **aren’t supposed to do**

## Cases

### The Constraint

#### Pseudo-Proof of Argument 3

We will use a proof by cases based on premise one. **Case 1:** Suppose  $(Cube(c) \wedge Small(c))$ . Then  $Small(c)$  follows. **Case 2:** Suppose  $Tet(c) \wedge Small(c)$ . Then  $Small(c)$  follows. So,  $Small(c)$  follows in either case. **But in case 1 we had  $Cube(c)$  and in case 2 we had  $Tet(c)$ , hence our conclusion follows:  $Small(c) \wedge Cube(c) \wedge Tet(c)$ .**

- Where exactly does this proof go wrong?
  - We picked a claim out of a case **after it was finished**
  - The assumptions and conclusions of a case are only available **within that case**

#### The Moral

What happens in a case, stays in a case.

## Temporary Assumptions

### The Constraints

- In proof by contradiction, like in proof by cases, we make a temporary assumption:
  - We assume  $P$  and try to show  $\perp$
  - But  $P$  is a **temporary assumption**
  - So anything we infer from it is also **temporary**
- Once we show  $\perp$ , we **discharge** the assumption of  $P$
- This temporary assumption of  $P$ , and the things we infer from it, corresponds to a subproof
- Once this assumption is **discharged**, we can’t reach back into the subproof

# Subproofs

## Drawing the Connections

### Proof with A Subproof

$$\begin{array}{l} \vdots \\ | \\ | \quad A \\ | \quad \vdots \\ | \quad B \\ | \\ \vdots \\ C \end{array}$$

- A subproof involves a **temporary assumption**
  - Like proof by contradiction
  - Like proof by cases
- So you can't reiterate lines from the subproof outside of the subproof

# Subproofs

## Guidelines for Use

### Guidelines for Using Subproofs

- 1 Once a subproof has ended, you can **never** cite one of its lines individually for any purpose, although you may cite the subproof as a whole (as in  $\vee$  **Elim** &  $\neg$  **Intro**)
- 2 In justifying a step of a proof, you may cite any earlier line of the **main proof**, or any subproof that **has not ended**

Let's do exercise 6.17 to solidify these points

# Proof Strategies

### How to Approach a Formal Proof

- 1 Understand what the sentences are saying
- 2 Decide whether you think the conclusion follows from the premises
- 3 If you don't think so, try to find a counterexample
- 4 If you do think so, try to give an informal proof
- 5 Use this informal proof to guide your formal proof
- 6 If you get stuck try working backwards

# Summary

## Negation

- We learned two new negation rules:  $\neg$  **Intro**,  $\neg$  **Elim**
- $\neg$  **Intro** mirrors the proof by contradiction method
- To mimic this method in  $\mathcal{F}$  we introduced the  $\perp$  symbol and two rules for it:  $\perp$  **Intro**,  $\perp$  **Elim**
- Proof by contradiction isn't just good for proving negated claims
  - It can also be used to prove positive claims

# Summary

## Subproofs & Strategy

- Mastering  $\mathcal{F}$  involves mastering subproofs
- Just like cases and reductio assumptions, there are constraints on how you can use subproofs
- We learned these constraints and the perils the guard us against
- We also learned how to approach proofs
  - There's strategy to it!
  - **Don't** just try to shuffle symbols!