Formal Proofs & Boolean Logic I:
Extending $\mathcal{F}$ with rules for $\wedge$ and $\vee$

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Announcements
For 09.27

❶ HW4 is due today
❷ HW1 grades are posted on Bb
   • Check on them!
❸ HW1-3 will be returned soon
   • After you have a look at them, please ask questions about grading

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Today's Class
Formal Proofs for Conjunction & Disjunction

• We'll be extending $\mathcal{F}$ w/rules for two of the Booleans:
   • Conjunction: $\wedge$ **Intro**, $\wedge$ **Elim**
   • Disjunction: $\vee$ **Intro**, $\vee$ **Elim**
• These formal rules will correspond to the informal inference steps and proof methods we discussed last class
   • Just like = **Elim** (formal) corresponded to the **Identity of Indiscernibles** (informal)
• We'll review the informal rules as we introduce their formal counterparts

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Outline

❶ Review

❷ Conjunction

❸ Disjunction

❹ Conclusion

Formal Proofs
In Review

- Today we are going to be doing formal proofs involving $\neg, \wedge, \vee$
- A while back we learned a bit about formal proofs
- Let's review the highlights

Methods of Proof
Two Varieties

- A proof is a step-by-step demonstration that some conclusion C is true whenever some premises $P_1, \ldots, P_n$ are true
- There are two ways of writing down these demonstrations
  1. **Informal Proof**: written up as a paragraph in ordinary language
  2. **Formal Proof**: written in an artificial language & formatted using special, visually suggestive notation
- Both ways are useful and have certain advantages
- In informal proofs we follow certain **inference steps** and **methods of proof**
- Similarly, formal proofs utilize **rules of inference**

Rules of Inference
Summary

= Elim

$$
\begin{array}{l}
\quad P(n) \\
\quad \vdots \\
\quad n = m \\
\quad \vdots \\
\rhd \quad P(m)
\end{array}
$$

Indiscernibility of Identicals

If n is m, then whatever is true of n is also true of m (where 'n' and 'm' are names)

- = **Elim** restates Ind. of Id.'s formally:
  - If you have a formula of the form $n = m$ and one of the form $P(n)$ then you can infer one of the form $P(m)$

Rules of Inference
Summary

= Introduction (= Intro)

$$\rhd \quad n = n$$

- Everything is self-identical
- You can reuse claims

Reiteration (Reit)

$$
\begin{array}{l}
\quad P \\
\quad \vdots \\
\rhd \quad P
\end{array}
$$

Formal Proof
An Example

$$
\begin{array}{c|ll}
1 & \text{SameSize}(a, b) & \\
2 & b = c & \\
3 & c = d & \\
\hline
4 & \text{SameSize}(a, c) & = \textbf{Elim}: 1, 2 \\
5 & \text{SameSize}(a, d) & = \textbf{Elim}: 3, 4 \\
\end{array}
$$

Formal Proofs
Generally Speaking

A Formal Proof

$$
\begin{array}{l|l}
P_1 & \\
\vdots & \\
P_n & \\
C_1 & \text{Justification } 1 \\
\vdots & \vdots \\
C_m & \text{Justification } m \\
C & \text{Justification } m + 1 \\
\end{array}
$$

- $P_1 - C$ are in FOL
- Premises: $P_1 - P_n$
- Conclusion: C
- Intermediate Conclusions: $C_1 - C_m$
- Justifications indicate where & how the formula on that line is being inferred
  - That is: from which formula(e) & by what rule of inference

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∧ Intro
From Formal to Informal

Conjunction Introduction

If you have proven (or have as premises) both P and Q, you can infer $P \wedge Q$

Example Informal Proof

We are given that a is a cube but we are also given that a is small. So it clearly follows that a is small and a cube.

- In a formal proof you **must** cite ∧ **Intro**
- Order does not matter

∧ Intro

$$
\begin{array}{c|l}
 & P_1 \\
 & \vdots \\
 & P_n \\
\triangleright & P_1 \wedge \ldots \wedge P_n \\
\end{array}
$$

Example Formal Proof

$$
\begin{array}{c|ll}
1 & \text{Cube}(a) & \\
2 & \text{Small}(a) & \\
3 & \text{Cube}(a) \wedge \text{Small}(a) & \wedge \textbf{Intro}: 1, 2 \\
\end{array}
$$

∧ Intro
An Example Formal Proof

$$
\begin{array}{c|ll}
1 & \text{Larger}(a, b) & \\
2 & b = c & \\
3 & \text{Cube}(a) & \\
4 & \text{Tet}(c) & \\
\hline
5 & \text{Larger}(a, c) & = \textbf{Elim}: 1, 2 \\
6 & \text{Tet}(c) \wedge \text{Larger}(a, c) & \wedge \textbf{Intro}: 4, 5 \\
7 & \text{Larger}(a, c) \wedge \text{Tet}(c) & \wedge \textbf{Intro}: 5, 4 \\
8 & \text{Larger}(a, c) \wedge \text{Tet}(c) \wedge b = c & \wedge \textbf{Intro}: 7, 2 \\
\end{array}
$$

✓ Goal: $\text{Larger}(a, c) \wedge \text{Tet}(c) \wedge b = c$

∧ Elim
From Informal to Formal

Conjunction Elimination

❶ From P ∧ Q you can infer P

❷ From P ∧ Q you can infer Q

An Example Informal Proof

a is both a cube and larger than *b*. So it is obvious that *a* is a cube.

- In a formal proof you **must** cite ∧ **Elim**

∧ Elim

$$P_1 \wedge \ldots \wedge P_n$$
$$\vdots$$
▷ P_i

Where $1 \leq i \leq n$

- P_i is any one of the conjuncts

Example Formal Proof

1	Cube(a) ∧ Larger(a, b)	
2	Cube(a)	∧ **Elim**: 1
3	Larger(a, b)	∧ **Elim**: 1

∧ Elim
An Example Formal Proof

1	Smaller(a, b) ∧ b = c ∧ Tet(c)	
2	Smaller(a, b)	∧ **Elim**: 1
3	b = c	∧ **Elim**: 1
4	Smaller(a, c)	= **Elim**: 2, 3
5	b = c ∧ Tet(c)	∧ **Elim**: 1
6	Smaller(a, c) ∧ b = c ∧ Tet(c)	∧ **Intro**: 4, 5

✓ Goal: Smaller(a, c) ∧ b = c ∧ Tet(c)

∨ Intro
From Informal to Formal Proof

Disjunction Introduction

If you have proven (or have as a premise) P, you can infer P ∨ Q

Example Informal Proof

We are given that *a* is a cube, so it must be the case that *a* is either a cube or small.

- From P_i you can infer any disjunction containing P_i
- It does matter which disjunct P_i is

∨ Intro

$$P_i$$
$$\vdots$$
▷ $P_1 \vee \ldots \vee P_i \vee \ldots \vee P_n$

Example Formal Proof

1	Cube(a)	
2	Cube(a) ∨ Small(a)	∨ **Intro**: 1

∨ Intro
An Example Formal Proof

1	Tet(a)	
2	Cube(e) ∧ Small(e)	
3	Tet(a) ∨ a = d	∨ **Intro**: 1
4	(Cube(e) ∧ Small(e)) ∨ Tet(c)	∨ **Intro**: 2
5	(Tet(a) ∨ a = d) ∧ ((Cube(e) ∧ Small(e)) ∨ Tet(c))	∧ **Intro**: 3, 4

✓ Goal: (Tet(a) ∨ a = d) ∧ ((Cube(e) ∧ Small(e)) ∨ Tet(c))

∨ Elim
From Informal to Formal Proof

Proof by Cases (Disjunction Elimination)

To prove C from $P_1 \vee \ldots \vee P_n$ using this method, show C from each of $P_1, \ldots P_n$

Example Informal Proof

Suppose we are given one premise: $(\mathsf{Tet}(a) \wedge b = a) \vee (\mathsf{Small}(b) \wedge c = d)$, and want to show that $b = a \vee c = d$ follows. We will use a proof by cases. **Case 1**: Suppose $\mathsf{Tet}(a) \wedge b = a$. Then $b = a$, and so $b = a \vee c = d$, clearly follows. **Case 2**: Suppose $\mathsf{Small}(b) \wedge c = d$. Then $c = d$, and so $b = a \vee c = d$, follows. In either case the conclusion follows.

∨ Elim

$P_1 \vee \ldots \vee P_n$

$\vdots$

$\quad P_1$

$\quad \vdots$

$\quad C$

$\vdots$

$\quad P_n$

$\quad \vdots$

$\quad C$

$\vdots$

$\triangleright \quad C$

∨ Elim
An Example Formal Proof

1	$(\mathsf{Tet}(a) \wedge b = a) \vee (\mathsf{Small}(b) \wedge c = d)$	
2	$\mathsf{Tet}(a) \wedge b = a$	
3	$b = a$	$\wedge$ **Elim**: 2
4	$b = a \vee c = d$	$\vee$ **Intro**: 3
5	$\mathsf{Small}(b) \wedge c = d$	
6	$c = d$	$\wedge$ **Elim**: 5
7	$b = a \vee c = d$	$\vee$ **Intro**: 6
8	$b = a \vee c = d$	$\vee$ **Elim**: 1, 2-4, 5-7

✓ Goal: $b = a \vee c = d$

∨ Elim
Subproofs

- The ∨ **Elim** rule makes use of some new notation in $\mathcal{F}$
- These are called subproofs
- The notation and name are designed to indicate that you have a mini-proof happening within a larger proof
- We will learn more about subproofs next class

∨ Elim

$P_1 \vee \ldots \vee P_n$

$\vdots$

$\quad P_1$

$\quad \vdots$

$\quad C$

$\vdots$

$\quad P_n$

$\quad \vdots$

$\quad C$

$\vdots$

$\triangleright \quad C$

∨ Elim
Another Example Formal Proof

We gave an informal proof for this argument last class:

$\mathsf{Cube}(a) \vee \mathsf{Smaller}(a, b)$

$\neg\mathsf{Cube}(a) \vee \mathsf{Smaller}(a, c)$

$\mathsf{Smaller}(b, c)$

$\mathsf{Smaller}(a, c)$

Proof: We use the proof by cases method:

❶ Suppose $\mathsf{Cube}(a)$. By the second premise we know that either $\mathsf{Cube}(a)$ is false or $\mathsf{Smaller}(a, c)$. So, it must be the case that $\mathsf{Smaller}(a, c)$

❷ Suppose $\mathsf{Smaller}(a, b)$. We are given that $\mathsf{Smaller}(b, c)$ and $\mathsf{Smaller}(\,,\,)$ is transitive, so $\mathsf{Smaller}(a, c)$

Let's construct a formal version of this proof in Fitch!

∨ Elim
In Class Exercise

*Construct a **formal proof** in F of the following argument. You will need to use ∨ **Elim**.*

(So far, F has the following rules:

= **Intro**, = **Elim**, ∧ **Intro**, ∧ **Elim**, ∨ **Intro**, ∨ **Elim**.)

| A ∨ (C ∧ B)
| B ∨ A

∨ Elim
Yet Another Example

Last class, you all wrote an informal proof for this argument:

| Smaller(a, c) ∨ FrontOf(a, b)
| Larger(a, c) ∨ BackOf(b, a)
| Between(c, a, b)
| FrontOf(a, b)

Proof: We will do a proof by cases based on the first premise.

❶ **Case 1**: Suppose Smaller(a, c). Then ¬Larger(a, c), since the two predicates are inverses. So it follows from premise two that BackOf(b, a), since at least one of the disjuncts must be true. But BackOf(,) and FrontOf(,) are also inverses, so it follows that FrontOf(a, b).

❷ **Case 2:** Suppose FrontOf(a, b). Well then FrontOf(a, b) follows.

Let's construct a formal version of this proof in Fitch

Even More Practice
Some More Exercises

Let's do exercises **6.3** & **6.5**

Summary
09.26

- We added rules to F for conjunction: ∧ **Intro**, ∧ **Elim**
- We added rules for disjunction: ∨ **Intro**, ∨ **Elim**
  - ∨ **Elim** corresponded to the proof by cases method
- We learned the new notation of subproofs in F:
  - Subproofs in F are like the cases in the proof by cases method