Methods of Proof for Boolean Logic

Proof by Contradiction

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Announcements

For 09.22

1. HW1 grades will be on Bb by end of week
2. HW4 is due on Tuesday
   - This one is mostly written
   - Feel free to type it out!
3. If you have problems with software bugs, try using version 2.7, which is available on Bb

Outline

The Big Picture

Where is Today?

- Logic is mainly about *logical consequence*
  - It's about conclusions following (or not following) from premises
- So far, we've explored two methods for understanding logical consequence:
  1. Proof
  2. Tautological Consequence (Ch.4: truth tables)
- But we've only considered informal proofs for $\wedge$ and $\vee$
- Today: we'll learn about informal proofs with $\neg$

The Plan
For Today's Class

1. Review the basics of proofs
2. Review informal proofs with $\wedge$ and $\vee$
3. Think about how to do informal proofs with $\neg$
4. Do some informal proofs with $\neg$
5. Try combining our strategies!

The Big Picture
But Wait...

- We had to learn truth tables, why proofs too?
- Truth tables are useful for the Booleans, but have significant limitations:
  1. **Impractical**: Truth tables get extremely large. An interesting argument could have well over 14 atomic sentences, the table would be over 16,000 rows!
  2. **Limited Applications**: as we learned Thursday, there are logical consequences that aren't tautological consequences. Why? Truth tables are blind to the logic of expressions other than the Booleans.
- The methods of proof fill this gap admirably

Proof
What is it?

Proof

A proof is a step-by-step demonstration which shows that a conclusion C must be true in any circumstance where some premises, say P_1 and P_2 are true

1. The step-by-step demonstration of C can proceed through intermediate conclusions
2. It may not be obvious how to show C from P_1 and P_2, but it may be obvious how to show C from some other claim Q that **is** an obvious consequence of P_1 and P_2
3. Each step provides conclusive evidence for the next

Proof
Steps, What?

The Nature of Steps

Each step of a proof appeals to certain facts about the meaning of the vocabulary involved. These facts are what we implicitly appeal to when we say a step is *obvious*.

- What kind of facts?
- Facts which guarantee that the step will never lead us from something true to something false
- Facts about the meaning of the words involved
- So for the Booleans we can use truth tables to guide us

Conjunction
From Meaning to Proof Rules

Truth Table for ∧

P	Q	P ∧ Q
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Conjunction Elimination

1. From P ∧ Q you can infer P
2. From P ∧ Q you can infer Q

Conjunction Introduction

1. From P and Q you can infer P ∧ Q

Disjunction
From Meaning to Proof Rules

Truth Table for ∧

P	Q	P ∨ Q
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

Disjunction Introduction

1. From P you can infer P ∧ Q

Proof by Cases
The Basics

- The first method of proof we learned about was called proof by cases
- It allowed us to use disjunctions to prove things
- Let's first look at an example where a disjunction is used to prove something

Proof by Cases
An Example

Disjunctions in a Proof

On a dare last night, Frank the frat boy hit himself in the eye with a wrench or a golf club.If he hit himself with a wrench he won't be able to play beer pong tonight. If hit himself with a golf club he'll also be unable to play beer pong tonight. So either way he'll end up sitting out tonight's beer pong session.

- Our first premise was a disjunction
- We reasoned that if the first disjunct was true, Frank couldn't play
- We then reasoned that if the second was true, Frank couldn't play
- We concluded that Frank can't play tonight

Proof by Cases
Official Version

So, more abstractly our strategy was this:

Proof by Cases (Disjunction Elimination)

To prove C from $P_1 \vee \ldots \vee P_n$ using this method, show C from each of $P_1, \ldots P_n$

- From our disjunctive premise we know at least one disjunct is true
- So showing that the truth of any one of them guarantees the truth of C, suffices to show that C follows from our disjunctive premise

Proof by Cases
Another Example

Claim: the following argument is valid

$$
\begin{array}{|l}
\mathsf{Cube}(a) \vee \mathsf{Smaller}(a,b) \\
\neg\mathsf{Cube}(a) \vee \mathsf{Smaller}(a,c) \\
\mathsf{Smaller}(b,c) \\
\hline
\mathsf{Smaller}(a,c)
\end{array}
$$

Proof: Given the first premise, we'll try a proof by cases:

1. Suppose Cube(a). By the second premise we know that either Cube(a) is false or Smaller(a, c). By assumption, Cube(a) is true. So, it must be the case that Smaller(a, c)
2. Suppose Smaller(a, b). We are given that Smaller(b, c) and Smaller(,) is transitive, so Smaller(a, c)

We've shown that in either case Smaller(a, c) follows

Proof by Contradiction
Proving a Negated Claim

- Okay, we've done proofs for conjunction and disjunction, but what about negation?
- We know one way of eliminating negation: $\neg\neg P \Leftrightarrow P$
- But, how would you go about proving a negated claim (¬ introduction), like ¬Cube(a)?
- Well, ¬Cube(a) is true if and only if Cube(a) is false
- So, prove ¬Cube(a) by showing that Cube(a) is false!
- But how do we do that?

Proof by Contradiction
Proving Something False

- There is a very important method for proving something false
  - Proof by Contradiction
  - A.k.a **Indirect Proof**, **Reductio ad Absurdum**

The Basic Idea of Proof by Contradiction

To show that P is false, it suffices to show that something which cannot possible be true, i.e. a contradiction, follows from P

Proof by Contradiction

What is a *Contradiction*?

Contradiction

- Intuitively, a contradiction is any sentence that cannot possibly be true, or any group of sentences that cannot all be true **simultaneously**
- The symbol $\perp$ is often used as a short-hand way of saying that a contradiction has been obtained

- Examples:
  1. $\neg\text{Cube}(a) \wedge \text{Cube}(a)$
  2. $a = b, b = c, a \neq c$
  3. $\text{SameCol}(a, b), \text{LeftOf}(a, b)$

Proof by Contradiction

What is It?

Proof by Contradiction (Preliminary Version)

To prove that P is false, show that a contradiction $\perp$ follows from P

Proving a Negated Claim

To prove $\neg$P, assume P and prove a contradiction $\perp$

- Contradictions are impossible, so false
- If you can show that P leads to a contradiction, then P must be false
- But if P is false, then $\neg$P must be true

Proof by Contradiction

A Simple Example

Claim: This argument is valid

$\neg\text{SameShape}(a, b)$

$b = c$

$\neg a = c$

Proof: We want to show $\neg a = c$ from the premises, so we will use a proof by contradiction

1. Suppose $a = c$
2. Then, from premise one $\neg\text{SameShape}(c, b)$ follows by Indiscernibility of Identicals
3. But by premise two, we know $\text{SameShape}(c, b)$. This is a contradiction, $\perp$!
4. So our supposition must have been false; that is, $\neg a = c$ must be true given the premises

Proof by Contradiction

Official Version

Proof by Contradiction (Official Version)

1. To prove that P is false, show that a contradiction $\perp$ follows from P
2. To prove that P is true, show that a contradiction $\perp$ follows from $\neg$P

- Proof by contradiction can also be used for proving un-negated claims

Proof by Contradiction

Another Example

Claim: This argument is valid

$$\begin{array}{|l} \mathsf{Tet(a)} \lor \mathsf{Large(a)} \\ \mathsf{Medium(a)} \\ \hline \neg\mathsf{Cube(a)} \end{array}$$

Proof: We want to show ¬Cube(a) from the premises, so we will use a proof by contradiction

1. Suppose Cube(a)
2. Then, from premise one Large(a) follows since *a* can't be a cube and a tetrahedron.
3. But by premise two, we know Medium(a)). This is a contradiction, ⊥!
4. So our supposition must have been false; that is, ¬Cube(a) must be true given the premises

Proof by Contradiction

In Class Exercise

Write an informal proof of this argument. Do a proof by contradiction.

$$\begin{array}{|l} \mathsf{Cube(b)} \lor \mathsf{LeftOf(b, c)} \\ \mathsf{SameCol(b, c)} \\ \hline \neg\mathsf{Tet(b)} \end{array}$$

Proof by Contradiction

Used to Prove an Un-negated Claim

$$\begin{array}{|l} \mathsf{b} = \mathsf{c} \land \mathsf{SameShape(c, a)} \\ \mathsf{Cube(b)} \\ \hline \mathsf{Cube(a)} \end{array}$$

Proof: We will use a proof by contradiction. First, suppose the conclusion is false, i.e. ¬Cube(a). By premise 1 we know that SameShape(c, a), so it must be that ¬Cube(c). But premise 1 tells us that b = c and together with premise 2 this entails that Cube(c), by the Indiscernibility of Identicals. This contradicts our early conclusion, so our supposition must have been false. Thus, the conclusion must be true when the premises are.

Proof by Contradiction

In Class Exercise

Write an informal proof of this argument, phrased in complete, well-formed English sentences. ***Hint***: *try a proof by contradiction. I encourage you to work in groups.*

5.15

$$\begin{array}{|l} \mathsf{Tet(b)} \\ \mathsf{Cube(c)} \\ \mathsf{Larger(c, b)} \lor \mathsf{c} = \mathsf{b} \\ \hline \mathsf{Smaller(b, c)} \end{array}$$

Proof by Contradiction

A More Advanced Example

Proof

We will do a proof by cases based on premise three:

1. Suppose ¬Small(d). We want to show d = c ∨ d = b, let's do this by indirect proof. This requires deriving ⊥ from the additional supposition that ¬(d = c ∨ d = b). First, note that by DeMorgan's Laws this implies d ≠ c and d ≠ b. From this and our first supposition, premise two clearly requires that ¬Dodec(d). But, from premise four and our original supposition, premise one clearly requires that Dodec(d). These requirements are contradictory, ⊥.
2. Now suppose d = b. Then d = c ∨ d = b follows immediately by disjunction intro

In either case d = c ∨ d = b follows, so the argument is valid

Argument 4

| Dodec(d) ∨ Tet(d) ∨ Small(d)
| ¬Dodec(d) ∨ (d = c) ∨ Small(d)
| ¬Small(d) ∨ d = b
| ¬Tet(d)
|---
| d = c ∨ d = b

- We had a proof by contradiction inside a proof by cases!
- This is like exercise 5.17!

Valid Arguments

What If The Premises are Inconsistent?

Logical Consequence, Validity

C is a logical consequence of $P_1, \ldots, P_n$ if and only if it is impossible for $P_1, \ldots, P_n$ to be true while C false

- Now think about an argument with inconsistent (contradictory) premises
  - Is it valid?
- Yes!
  - Why: it's impossible for the premises to be true
- So, it is impossible for the premises to be true while the conclusion is false!
- But, crucially, the argument is not sound

Summary

09.22

1. We remembered what proof by cases was
2. We learned a powerful new proof method:
   - Proof by contradiction
3. We learned that our two proof methods can be mixed
4. We wrapped our head around the fact that any argument with contradictory premises is valid
   - Importantly, though, such an argument is never sound