The Logic of Boolean Connectives

Truth Tables, Tautologies & Logical Truths

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Announcements

For 09.15.11

1. HW1 was due on Tuesday
   - If you didn't turn it in, it's late
2. HW2 & HW3 are due next Tuesday (09.20)
   - Electronic HW must be submitted before class
   - Written HW must be handed in at beginning of class
   - Otherwise, it's late
3. Optional sections have been scheduled
   - Wednesday 1:25-2:10, Uris 307
   - Thursday 1:25-2:10, Uris G22

Outline

Introduction

Truth Functions

- Last class, we learned the meaning of $\wedge, \vee, \neg$ in terms of truth functions
- We also saw that truth functions allowed us to do something useful:
  - Figure out the truth value of a complex sentence from the truth values of its atomic parts, and vice versa
- For example, we know that $\neg(\mathsf{Cube}(\mathsf{a}) \vee \mathsf{Cube}(\mathsf{b}))$ is true in a world where neither a nor b are cubes, since:
  - If $\neg(\mathsf{Cube}(\mathsf{a}) \vee \mathsf{Cube}(\mathsf{b}))$ is true, then $\mathsf{Cube}(\mathsf{a}) \vee \mathsf{Cube}(\mathsf{b})$ is false
  - If $\mathsf{Cube}(\mathsf{a}) \vee \mathsf{Cube}(\mathsf{b})$ is false, then $\mathsf{Cube}(\mathsf{a})$ and $\mathsf{Cube}(\mathsf{b})$ are false

Introduction
Truth Tables

- These calculations — ones like the one we just went through — are a bit clunky
- There's a more elegant method: Truth tables
- As it turns out, truth tables will also provide us with a helpful way to understand 3 core logical concepts:
  1. Logical Consequence
  2. Logical Truth
  3. Logical Equivalence
- Today, we'll learn all about truth tables & these applications!

Review
The Boolean Connectives

Truth Table for ¬

P	¬P
TRUE	FALSE
FALSE	TRUE

Truth Table for ∧

P	Q	P ∧ Q
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Truth Table for ∨

P	Q	P ∨ Q
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

- ¬ flips the value
- ∧ takes the 'worst' value
- ∨ takes the 'best' value

The Basics
Step 1: The Reference Columns

- We are going to construct a truth table for

  (1) Cube(a) ∨ ¬Cube(a)

First, some columns:

1. Reference Columns: columns for each atomic sub-sentence of (1)
2. A column for (1) itself

Truth Table for (1)

Cube(a)	Cube(a) ∨ ¬Cube(a)
T	
F	

Second, fill the reference columns w/truth values.

- One row for each unique logical possibility

The Basics
Step 2: Inside Out

Truth Table for (1)

Cube(a)	Cube(a)	∨	¬Cube(a)
T			F
F			T

- Third, fill column beneath innermost connective ¬:
  - In the first row, Cube(a) is T so ¬Cube(a) is F
  - In the second row, Cube(a) is F so ¬Cube(a) is T

The Basics

Step 3: The Main Connective

Truth Table for (1)

Cube(a)	Cube(a) $\lor$ $\neg$Cube(a)
T	T F
F	T T

- Last, fill columns beneath outermost connective $\lor$:
  - In the first row, Cube(a) is T and $\neg$Cube(a) is F, so their disjunction is T
  - In the first row, Cube(a) is F and $\neg$Cube(a) is T, so their disjunction is T
- This column lists every logically possible truth value for (1)

The Basics

Summary: In General

How to Construct a Truth Table for Any Sentence P

1. **Reference Columns**: Draw a column for each atomic sub-sentence of P, these columns are called the reference columns and are filled with every possible combination of truth-values for the sub-sentences
2. **Inside Out**: Draw a column for P itself. Then fill in the column below P's innermost connective. Repeat for the next innermost connective, until you get to the main connective.
3. **Main Connective**: Fill in the column under the main connective. This row lists the possible truth values of P

Reference Columns

There is More to It...

- When you have more than one atomic sub-sentence, filling in the reference columns requires more thought
- Remember that each row of the reference columns lists a unique logical possibility
- Also remember that there is supposed to be a row for every unique possibility
- Okay, well how many rows would we need for a formula with 2 atomic sub-sentences?

Reference Columns

How Many Rows?

Let's figure it out:

- We have 2 atomic sub-sentences
- Each can have 2 different truth values (T,F)
- So there $2^2 = 4$ possible combinations of truth values and atomic sub-sentences
- Therefore, a table for a formula with 2 atomic sub-sentences needs 4 rows

You Always Need 2^n Rows

In general, if there are n atomic sub-sentences of P then there will be 2^n possible assignments of truth values to those atomic sub-sentences, in which case the truth table for P should have 2^n rows.

Another Example

Step 1

- Let's construct a truth table for:
  (2) $\neg(\mathsf{Cube(a)} \wedge \mathsf{Cube(b)})$
- We need 2 reference columns:
  - In them, we need a row for each of the 4 logical possibilities
  - Cube(a) can be T and Cube(b) T; Cube(a) T and Cube(b) F, and so on.

Truth Table for (2)

Cube(a)	Cube(b)	¬(Cube(a) ∧ ¬Cube(b))
T	T	
T	F	
F	T	
F	F	

Another Example

Steps 2 & 3

Truth Table for (2)

Cube(a)	Cube(b)	¬(Cube(a)	∧	¬Cube(b))
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	T

- Next, the innermost connective ¬:
  - It will flip each value of Cube(b)
- Now, the next innermost ∧:
  - ∧ takes worst value of the pair
- Finally, the main connective ¬:
  - ¬ flips the value of the conjunction we just computed

Building Reference Columns

A Helpful Routine

- You need to list each possibility exactly once when filling in reference columns
- Here's a helpful routine for this:
  1. In the innermost ref. column, you alternate T's and F's
  2. In the next innermost column, you **double** that alternation, and so on for any more rows

Truth Table for (2)

Cube(a)	Cube(b)	¬(Cube(a) ∧ ¬Cube(a))
T	T	
T	F	
F	T	
F	F	

Let's put this routine to work

Yet Another Example

Step 1

We'll construct a table for:

(3) $(\mathsf{Cube(a)} \wedge \neg\mathsf{Cube(b)}) \vee \neg\mathsf{Cube(c)}$

Let $\mathsf{A} = \mathsf{Cube(a)}, \mathsf{B} = \mathsf{Cube(b)}, \mathsf{C} = \mathsf{Cube(c)}$

Table for (3)

A	B	C	(A ∧ ¬B) ∨ ¬C
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

- First, the 8 rows of the reference columns:
  1. Alternate on innermost column
  2. Double this alternation on the next
  3. Double again

Yet Another Example
Steps 2 & 3

Table for (3)

A	B	C	(A	∧	¬B)	∨	¬C
T	T	T		F	F		F	F
T	T	F		F	F		T	T
T	F	T		T	T		T	F
T	F	F		T	T		T	T
F	T	T		F	F		F	F
F	T	F		F	F		T	T
F	F	T		F	T		F	F
F	F	F		F	T		T	T

- Next, rows for innermost connectives ($\neg$)
  - $\neg$B: flips value of B
  - $\neg$C: flips value of C
- Now, the row for the next innermost connective ($\wedge$)
  - $\wedge$ takes lowest value
- Finally, the row for the main connective ($\vee$):
  - $\vee$ takes highest value

You Try It
In Class Exercise

Break into groups of 4-6 and construct a truth table for:

(4) $(\mathsf{B} \vee \neg \mathsf{C}) \wedge \neg \mathsf{A}$

Boole
An Introduction

- *LPL* contains a program called Boole, which is for constructing truth tables
- Now that you've done a one by hand, you can appreciate how nice of a tool this is!
- Let's run through the basics of Boole by using it to construct a table for (3)

Truth Tables
Summary

- Okay, we've learned how to draw these pretty tables, but how do they help us do logic?
  1. Truth tables probe logical possibility
  2. This underlies several important concepts:
     - Logical truth
     - Logical consequence
     - Logical equivalence
- Let's see how

Logical Truth
The Basics

Logical Truth

P is a logical truth if and only if it is **logically necessary**. That is, it is **not possible** for the laws of logic to hold while P is false.

- Logical truths are those sentences which are guaranteed by logic alone to be true
- Logical possibility is different from physical & other kinds of possibility
  - Cube(a) ∨ ¬Cube(a) vs. Cube(a) ∨ Tet(a) ∨ Dodec(a)
  - Traveling the speed of light vs. being a round square
- This sounds vague, can we do any better?
  - Yes, if we use truth tables

Tautologies

Tautology

P is a tautology if and only if the truth table for P has only T's in the column under P's main connective

Truth Table for (1)

Cube(a)	Cube(a)	∨	¬Cube(a)
T		T	F
F		T	T

- So, (1) is a tautology
- Intuitively, is (1) a logical truth?
  - Yes!
- So, it looks like the idea of a tautology is a way of making the idea of a logical truth a bit more precise
- This is because truth tables are a precise way of thinking about logical possibility

Tautologies
Some Examples

- Cube(c) ∨ ¬Cube(c) is a tautology
- ¬(Tet(a) ∧ ¬Tet(a)) is a tautology
- Tet(a) ∨ Cube(a) is not a tautology

Tautologies vs. Logical Truths

- Recall that logical truths are sentences which are guaranteed to be true by the laws of logic alone
- All tautologies are logical truths
- But are all logical truths tautologies?
  - In other words, can we just replace the idea of a logical truth with that of a tautology?
- No! Seeing this actually takes a little creativity

A Curious Logical Truth

Which isn't a Tautolgy

- Surely it is a logical truth that Jay is Jay and that Kay is Kay:

  (5) $\mathsf{j} = \mathsf{j} \wedge \mathsf{k} = \mathsf{k}$
- But consider the truth table for (5):

Truth Table for (5)

$\mathsf{j} = \mathsf{j}$	$\mathsf{k} = \mathsf{k}$	$\mathsf{j} = \mathsf{j} \wedge \mathsf{k} = \mathsf{k}$
T	T	T
T	F	F
F	T	F
F	F	F

- First, reference columns
- Second, main connective
- But wait, there are F's in that column!
- When you build reference columns, you just list all the combinations
- But, some combinations don't make sense!

Tautologies & Logical Truths

Summary

Remember

1. P is a tautology if and only if every row of the truth table assigns T to P
2. If P is a tautology, then P is a logical truth
3. But some logical truths are not tautologies
4. P is TT-possible if and only if at least one row of its truth table assigns T to P

Let's do exercise **4.5**

Summary

What We Just Did

- We learned how to construct truth tables, by hand & with Boole
- We then applied truth tables to the problem of precisely defining:
  1. Logical truth
- We came up with a similar concept:
  1. Tautology
- We saw that all Tautologies are logical truths, but some logical truths are not tautologies

Equivalence

Two Varieties

Logical Equivalence

Two sentences are logically equivalent if and only if they have the same truth values in every possible situation

- For example: $\mathsf{Tet}(\mathsf{a})$ and $\neg\neg\mathsf{Tet}(\mathsf{a})$ are logically equivalent

Tautological Equivalence

Two sentences are tautologically equivalent just in case the columns under their main connectives in a joint truth table are identical

- What is a joint truth table?

Equivalence
Joint Truth Tables

- The idea of a joint truth table is quite simple, just add a column on the right for another formula and calculate as before

Joint Truth Table

P	Q	¬	(P ∧ Q)	¬P	∨	¬Q
T	T	F	T	F	F	F
T	F	T	F	F	T	T
F	T	T	F	T	T	F
F	F	T	F	T	T	T

1. Ref. columns
2. Inner connectives
3. Main connectives

- The columns under the main connectives are identical
- So, these two sentences are tautologically equivalent

Equivalence
Logical vs. Tautological

- We started by characterizing two kinds of equivalence
  1. Logical
  2. Tautological
- Every two sentences that are tautologically equivalent are logically equivalent
- Does the reverse hold?
- No, this pair is logically equivalent:
  (6) $\mathsf{a = b \wedge Cube(a)}$
  (7) $\mathsf{a = b \wedge Cube(b)}$
- But we can show with Boole that they aren't tautologically equivalent

Consequence
Two Varieties Again

Logical Consequence

C is a logical consequence of $P_1, \ldots, P_n$ just in case it is logically impossible for C to be false while $P_1, \ldots, P_n$ are true

- We've already met this concept of validity/consequence
- It doesn't help us much with figuring out whether an argument is valid
- Proof provides one method, truth tables another:

Tautological Consequence

C is a tautological consequence of $P_1, \ldots P_n$ just in case every row in their **joint truth table** that lists T under $P_1, \ldots, P_n$ also lists T under C

Tautological Consequence
An Example

Argument 1

$A \vee B$
$\neg A$
B

Joint Truth Table for Argument 1

A	B	A ∨ B	¬A	B
T	T	T	F	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

- First two columns for the premises
- Last column for conclusion
- Every row where both premises are T, the conclusion is T
- So B is a tautological consequence of A ∨ B and ¬A

Tautological Consequence

Another Example: Exercise 4.20

Let's run through exercise 4.20

Tautological Consequence

In-Class Exercise

Exercise **4.21**

$$\begin{array}{|l} \mathsf{Taller(claire, max)} \lor \mathsf{Taller(max, claire)} \\ \mathsf{Taller(claire, max)} \\ \hline \neg\mathsf{Taller(max, claire)} \end{array}$$

Consequence

Questions

1. If C is a tautological consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$, is C a logical consequence of $\mathsf{P}_1, \ldots \mathsf{P}_n$?
   - Yes, clearly
2. If C is a logical consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$, is C a tautological consequence of $\mathsf{P}_1, \ldots \mathsf{P}_n$?
   - No, lets show it using Boole to construct a joint truth table for this argument:

Argument 2

$$\begin{array}{|l} \mathsf{a = b \land b = c} \\ \hline \mathsf{a = c} \end{array}$$

Taut Con

Tautological Consequence in Fitch

- Truth tables provide a powerful but purely mechanical procedure to test for logical consequence
- But, they often get really tedious and long
- But, that's what computers are good at
- Fitch has a built-in mechanism for testing for tautological consequence
  - **Taut Con**
- Much like **Ana Con**, this is not a rule of inference, but a computational mechanism
- Let's run through exercise 4.26

Summary

What we did today

- We learned how to construct truth tables, by hand & with Boole
- We applied truth tables to the problem of precisely defining:
  1. Logical truth
  2. Logical equivalence
  3. Logical consequence
- We came up with three similar concepts:
  1. Tautology
  2. Tautological equivalence
  3. Tautological consequence
- In each case Tautological implied Logical, but Logical did not imply Tautological