# The Boolean Connectives

## Negation, Conjunction & Disjunction

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# Announcements

For 09.13

1. HW1 is due now
   - 1.1-4 and 2.8 were due to the Grade Grinder
   - 2.2, 2.6, 2.8 should be submitted now on paper
2. HW2 & HW3 are due next Tuesday
3. We are trying to find space to hold the sections
   - Stay tuned!
4. Our TA (Theo Korzukhin) has an office hour
   - Tuesday: 1:25-2:25pm, Goldwin Smith 223
5. My office hours are now Thursdays from 2:40-3:40pm
   - Goldwin Smith 237





# Outline







# Today's Topic

Overview

- We've only looked at the logic of simple sentences like:

  *Mars is red*

  Red(mars)
- But more complicated sentences have equally interesting logical properties
  - Sentences in which simple ones are modified:
    - *It is not the case that Mars is red*
    - Etc.
  - Sentences which compound multiple simple ones:
    - *Bill is rad and Logan is bogus*
    - *Jay is home or Kay is home*
    - Etc.
- These all involve a connective of some sort

## Today's Topic

- Today we'll begin to learn about the logic of these connectives:

**The Boolean Connectives**

1. *not*, *it is not the case that* (Negation)
2. *and* (Conjunction)
3. *or* (Disjunction)

- Today we'll:
  - Learn a bit about what these words mean
  - Learn about some new symbols in FOL that are used to represent them
  - Discuss some issues these symbols raise
  - Discuss the connection between English sentences & FOL sentences with these symbols





## Connectives & Meaning

### Truth-Functions & Logic Games

- There are two ways we will explore the meaning of the Boolean connectives:
  1. Truth-Functions
  2. Logic Games
- Before getting into the details of todays topics, let's go over a few helpful general facts about these approaches





## Truth-Functions

### Parts & Wholes

- When considering the meaning of a declarative sentence one important thing to consider is its truth conditions
  - That is, the most general conditions under which it would be true

**Truth-functions**

The idea behind truth-functions is that the truth-value of a whole sentence can be computed solely from the truth value of it's simplest parts





## Truth-Functions

### The Boolean Connectives as Truth-Functions

- This idea has been applied to analyzing the meaning of the Boolean connectives
- For example, in the case of *and*:

  (1) *Kay ran and Jay ran*

  The truth value of (1) can be computed from the truth value of it's two simplest sentences

  (1a) *Kay ran*
  (1b) *Jay ran*
- The same approach can be pursued for Negation and Disjunction
- We will see these approaches in detail momentarily

# Truth-Functions

## Historical Notes

- The truth-functional approach to meaning originated with George Boole and Gottlob Frege, with important contributions along the way by Alfred Tarski
- It survives in modern philosophical theorizing about language and the mind:
  - Donald Davidson
  - Richard Montague
  - Jerry Fodor
  - And many others
- It also plays an important role among contemporary linguists who study meaning & communication





# Logic Games

## Meanings, Strategies & Games

- Another way of thinking about the meaning of a complex sentence draws on the idea of a game
- Imagine Jay and Kay disagree about the truth value of a complex sentence
  - They can resolve their disagreement by repeatedly challenging each other to justify their claims in terms of simpler claims, until finally their disagreement is reduced to a simple atomic claim
    - At that point, they can just examine the world to see who is right (ideally)
- These successive challenges can be thought of as a game where one player will win & one will loose
- On this approach, the meaning of a connective can be identified with the winning strategy for a game based on a sentence containing that connective





# Logic Games

## Historical Notes

- The game-theoretic approach to the meaning of the connectives originated with Jaakko Hintikka and Leon Henkin
  - It drew on philosophical ideas about mathematics, mind & language from:
    - Ludwig Wittgenstein (late work)
    - L.E.J. Brouwer
  - And it survives in more modern philosophical discussions of mathematics, mind & language:
    - Michael Dummett
    - Robert Brandom
    - Jaakko Hintikka
    - John Searle
    - And many others





# Negation

## English & FOL

- In English, negation is expressed in a number of ways:
  1. *not*
  2. *it is not the case that...*
  3. *non-*
  4. *un-*
- In FOL negation is expressed with one symbol: $\neg$
- $\neg$ attaches to the front of any formula to produce a new one:
  - ¬Bizarre(jay)
  - ¬SameSize(a, b)
  - ¬¬SameSize(a, b)

# Negation
## English vs. FOL

- The grammar for ¬ is sometimes the same as negation in English:
  - *It is not the case that Jay is bizarre* ⇝ ¬Bizarre(jay)
- But it can be very different:
  - *Jay is not bizarre* ⇝ ¬Bizarre(jay)
  - *Kay is unpopular* ⇝ ¬Popular(kay)
- Okay, but what does negation **mean**?





# The Meaning of Negation
## Truth-Functions & Games

**Truth-Table for ¬**

| P | ¬P |
|---|---|
| TRUE | FALSE |
| FALSE | TRUE |

- When P is true, ¬P is false
- When P is false, ¬P is true
- The truth value of ¬P is a function of P's truth value

**Game Rule for ¬**

1. If you commit to the truth of ¬P, you are committed to the falsity of P
2. If you commit to the falsity of ¬P, you are committed to the truth of P

Let's absorb this a bit more w/the **You try it** from §3.1





# Conjunction
## English & FOL

- In English, conjunction is expressed in a number of ways:
  1. *and*
  2. *moreover*
  3. *but*
- In FOL, conjunction is expressed with one symbol: ∧
- ∧ connects two sentences of FOL to form a new one:
  - Large(a) ∧ Cube(a)
  - Large(a) ∧ ¬Cube(a)
  - Large(a) ∧ ¬Cube(a) ∧ Small(b)





# Conjunction
## English vs. FOL

- The grammar for ∧ is sometimes the same as conjunction in English:
  - *D.M.C is loud and Jam-Master Jay is proud*

    ⇝ Loud(dmc) ∧ Proud(jmj)
- But it is usually **very** different:
  - *Brittany is deranged and delirious*

    ⇝ Deranged(brittany) ∧ Delirious(brittany)
  - *Bill and Ted had an excellent adventure*

    ⇝ ExAdventure(bill) ∧ ExAdventure(ted)

# The Meaning of Conjunction

## Truth-Functions & Games

**Truth-Table for ∧**

| P | Q | P ∧ Q |
|---|---|---|
| TRUE | TRUE | TRUE |
| TRUE | FALSE | FALSE |
| FALSE | TRUE | FALSE |
| FALSE | FALSE | FALSE |

- P ∧ Q is true when P is true and Q is true
- Otherwise, P ∧ Q is false
- The truth value of P ∧ Q is a function of the truth values of P and Q

**Game Rule for ∧**

1. If you commit to the truth of P ∧ Q, you commit to the truth of both P and Q
2. If you commit to the falsity of P ∧ Q, you commit to the falsity of either P or Q

Let's solidify this w/Exercise **3.5**





# Disjunction

## English & FOL

- In English, disjunction is expressed with *or*
- In FOL, disjunction is expressed with ∨
- ∨ connects two sentence of FOL to form a new one:
  - Cube(a) ∨ Tet(a)
  - Cube(a) ∨ ¬Tet(a)
  - Cube(a) ∨ ¬Tet(a) ∧ Small(a) ∨ ¬Large(a)





# Disjunction

## English vs. FOL

- The grammar for ∨ is sometimes the same as disjunction in English:
  - *Mexico is beautiful or I drank too much tequila*
    $\rightsquigarrow$ Beautiful(mexico) ∨ 2muchTequila(ws)
- But it is often **very** different:
  - *Bill will pass or fail*
    $\rightsquigarrow$ Pass(bill) ∨ Fail(bill)
  - *Bill or Ted will party*
    $\rightsquigarrow$ Party(bill) ∨ Party(ted)





# The Meaning of Disjunction

## Truth-Functions & Games

**Truth-Table for ∨**

| P | Q | P ∨ Q |
|---|---|---|
| TRUE | TRUE | TRUE |
| TRUE | FALSE | TRUE |
| FALSE | TRUE | TRUE |
| FALSE | FALSE | FALSE |

- P ∨ Q is false when both P and Q are false
- Otherwise, P ∨ Q is true
- The truth value of P ∨ Q is a function of the truth values of P and Q

**Game Rule for ∨**

1. If you commit to the truth of P ∨ Q, you must commit to the truth of either P or Q
2. If you commit to the falsity of P ∨ Q, you commit to the falsity of both P and Q

Let's solidify this w/Exercise **3.8**

# Summary

## The Booleans

**Summary**

1. Negation (¬) flips truth values
2. Conjunction (∧) takes the worst of the two truth values
3. Disjunction (∨) takes the best of the two truth values





# Ambiguity

## What?

(2) *Max is home or Claire is home and Carl is happy*

- (2) is ambiguous (it has multiple interpretations):
  a. *Max or Claire is home and Carl is happy*
  b. *Either Max is home, or Claire is home and Carl is happy*
- You might try to translate (2) into FOL as:

  (3) Home(max) ∨ Home(claire) ∧ Happy(carl)
- (3) is just as ambiguous as (2)!
- FOL aims to eliminate this ambiguity





# Ambiguity

## Introducing Parentheses

(3) ∗ Home(max) ∨ Home(claire) ∧ Happy(carl)

- FOL outlaws ambiguity by considering sentences like (3) ungrammatical
  - (3) is like a sentence fragment in English
- To make (3) grammatical parentheses can be added in two ways:

  (3a) (Home(max) ∨ Home(claire)) ∧ Happy(carl)

  (2a) *Max or Claire is home and Carl is happy*

  (3b) Home(max) ∨ (Home(claire) ∧ Happy(carl))

  (2b) *Either Max is home, or Claire is home and Carl is happy*





# Parentheses Matter

## Non-Equivalence

- Parentheses matter to meaning
  - These two sentences are not equivalent:

    (4) Tet(a) ∨ (Tet(b) ∧ Tet(c))

    (5) (Tet(a) ∨ Tet(b)) ∧ Tet(c)
  - These two aren't equivalent either:

    (6) ¬(Large(a) ∨ Large(b))

    (7) ¬Large(a) ∨ Large(b)
- Let's see this in more detail by doing exercise **3.12**

## Parentheses
### The Analogy With Arithmetic: Grouping Can Matter

- Different groupings w/parentheses in FOL can yield different truth conditions
  - This is just like orders of operation in arithmetic:

| | |
|---|---|
| $2 + (3 \times 4) = 14$ | $\text{Tet(a)} \land (\text{Tet(b)} \lor \text{Tet(c)})$ |
| $(2 + 3) \times 4 = 20$ | $(\text{Tet(a)} \land \text{Tet(b)}) \lor \text{Tet(c)}$ |
| Different! | Different! |
| $-(2 + 3) = -5$ | $\neg(\text{Tet(a)} \land \text{Tet(b)})$ |
| $-2 + 3 = 1$ | $\neg\text{Tet(a)} \land \text{Tet(b)}$ |
| Different! | Different! |





## Parentheses
### The Analogy With Arithmetic: Sometimes Grouping Doesn't Matter

- But, just like arithmetic sometimes grouping does not matter:

- $+$ is associative:
  - $2 + (3 + 4) = 9$
  - $(2 + 3) + 4 = 9$
  - Same!
- $\land$ is too:
  - $\text{Tet(a)} \land (\text{Tet(b)} \land \text{Tet(c)})$
  - $(\text{Tet(a)} \land \text{Tet(b)}) \land \text{Tet(c)}$
  - Same!
- $\times$ is associative:
  - $2 \times (3 \times 4) = 24$
  - $(2 \times 3) \times 4 = 24$
  - Same!
- $\lor$ is too:
  - $\text{Tet(a)} \lor (\text{Tet(b)} \lor \text{Tet(c)})$
  - $(\text{Tet(a)} \lor \text{Tet(b)}) \lor \text{Tet(c)}$
  - Same!





## Parentheses
### Official Policy

**Our Policy on Parentheses**

Parentheses must be used whenever ambiguity would result from their omission. In practice, this means that conjunctions & disjunctions must be 'wrapped' in parentheses whenever combined by means of some other connective

- This allows us to omit parentheses when unnecessary, but requires us to include them when they are!
- This will make our formulas look as simple as possible





## Equivalences
### DeMorgan's Laws & Double Negation

- Just like any language, there are many ways to say the same thing in FOL
- Here are three you should be aware of

**Important FOL Equivalences**

1. **Double Negation**: $\neg\neg P \Leftrightarrow P$
2. **DeMorgan**: $\neg(P \land Q) \Leftrightarrow \neg P \lor \neg Q$
3. **DeMorgan**: $\neg(P \lor Q) \Leftrightarrow \neg P \land \neg Q$
4. There are many others we'll discover along the way

Let's examine DeMorgan's Equivalences with exercise **3.16**

# Translation
## An Outline

- Translating from English to FOL is a useful ability
  - But, it can take some practice & skill
- Today we'll learn:
  1. When we will consider a translation to be correct
  2. Some tricks for translating conjunctions, disjunctions & negations





# Translation
## When is One Correct?

**Correctness Condition for Translations**

In order for a FOL sentence to be a good translation of an English sentence, it is sufficient that the two sentences have the same truth values in all possible circumstances, that is, that they have the same truth conditions

- Note that it is not sufficient for the two sentences have the same truth value in some particular world or situation





# Translation
## Varieties of Conjunction

- There are fine-grained differences between *and*, *but*, *however*, *yet* & *nonetheless*
- But because we are only interested in truth conditions they will all be translated as $\wedge$
- So, for example (8)-(12):

(8) *Jay is large and Kay is in charge*

(9) *Jay is large but Kay is in charge*

(10) *Jay is large however Kay is in charge*

(11) *Jay is large yet Kay is in charge*

(12) *Jay is large, Kay is in charge*

Are all translated as:

(13) $\mathsf{Large(jay)} \wedge \mathsf{InCharge(kay)}$





# Translation
## Both

- *Both* is often used to clarify where exactly the conjunction is:
  - A good translation of:

    (14) *It is not true that Claire and Max are both at home*

    Is:

    (15) $\neg(\mathsf{Home(claire)} \wedge \mathsf{Home(max)})$

    As opposed to:

    (16) $\neg\mathsf{Home(claire)} \wedge \mathsf{Home(max)}$
- A similar device exists for disjunction

# Translation
## Either Or

- *Either... or* acts like our parentheses in FOL:
- Consider:

(17) *Either* ***a*** *is small and a cube or it is large*

A good translation of this is:

(18) $(\mathsf{Small}(a) \wedge \mathsf{Cube}(a)) \vee \mathsf{Large}(a)$

As opposed to:

(19) $\mathsf{Small}(a) \wedge (\mathsf{Cube}(a) \vee \mathsf{Large}(a))$





# Translation
## Neither Nor

- Another common construction involving disjunction is *neither... nor*
- These sentences can be translated as *not... or*, with *neither* marking the beginning of the disjunction
- Consider:

(20) *Neither Jay nor Kay is home*

A good translation of this is:

(21) $\neg(\mathsf{Home}(jay) \vee \mathsf{Home}(kay))$

As opposed to:

(22) $\neg\mathsf{Home}(jay) \vee \mathsf{Home}(kay)$





# Translation
## Summary

1. We will consider a translation is correct when it correctly captures the target sentence's truth conditions
2. There are a variety of negations, all translated using $\neg$
3. There are a variety of conjunctions, all of which translate using $\wedge$
   - *and*, *but*, *however*, *yet* & *nonetheless*
4. To translate *or*, use $\vee$
5. *Either* and *both* are used to indicate grouping

To solidify, we'll do parts of exercises **3.21** & **3.22**





# Summary
## Today's Class

- We met the Booleans:
  1. **Conjunction**: *and*, $\wedge$
  2. **Disjunction**: *or*, $\vee$
  3. **Negation**: *not*, $\neg$
- We learned how parentheses in FOL make sure that $\neg, \vee, \wedge$ play nicely together
- We saw a few important equivalences
  - **Double Negation**: $\neg\neg P \Leftrightarrow P$
  - **DeMorgan**: $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
    **DeMorgan**: $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

# Summary

## Today's Class (Cont'd)

- We learned a tricks for translating from English to FOL
- Correct translations capture truth conditions
- Various important facts about translating conjunctions, disjunctions & negations:
    - There are many forms of conjunction in English, but all get translated as $\wedge$
    - *Both* is used to clarify the location of a conjunction
    - *Either... or* does the same for disjunction
    - *Neither... nor* is translated as *not... or*