

Formal Proofs for Quantifiers

\forall Intro and \forall Elim

William Starr

04.21.09

Announcements

For 04.21

- ① HW11 is due now
- ② HW12 will be due on **Thursday 04.30**
 - HW12 will be a practice final exam
- ③ There will be no HW13!
- ④ **Final Exam:**
 - Take-home portion given out on **Thurs 04.30**
 - Take-home portion due at in-class final exam on **Weds 05.13 (8-11am)**

Outline

- ① Review
- ② Formal Rules for \forall

Summary of Informal Steps & Methods

Overview

- For universal quantification, we've learned one informal inference step & two informal proof methods:
 - Universal Elimination
 - Universal Introduction
 - General Conditional Proof

Summary of Informal Steps & Methods

Universal Elimination (Official Version)

If you have $\forall x S(x)$, you may infer $S(c)$, provided that 'c' refers to an object in the domain of discourse

Universal Introduction

To prove $\forall x S(x)$:

- 1 Introduce a **new** name c to stand for a completely **arbitrary** member of the domain of discourse
- 2 Prove $S(c)$
- 3 Conclude $\forall x S(x)$

An Example

Using Universal Elimination & Introduction

- 1 $\forall x \text{Tet}(x)$
- 2 $\forall x (\text{Tet}(x) \vee \text{Small}(x))$

Proof: We will use universal introduction to prove the conclusion. Let c name an arbitrary block. From 1 $\text{Tet}(c)$ follows by universal elimination. But if *that* is true, so is $\text{Tet}(c) \vee \text{Small}(c)$ (by disjunction introduction). Since c was arbitrary, it follows that $\forall x (\text{Tet}(x) \vee \text{Small}(x))$.

Summary of Informal Steps & Methods

General Conditional Proof

General Conditional Proof

To prove $\forall x (A(x) \rightarrow B(x))$:

- 1 Introduce a **new** name c to stand for a completely **arbitrary** member of the domain of discourse assume $A(c)$
 - 2 Prove $B(c)$
 - 3 Conclude $\forall x (A(x) \rightarrow B(x))$
- This is equivalent to using universal introduction *and* conditional proof together

An Example

General Conditional Proof

- 1 $\text{Medium}(a)$
- 2 $\forall y (\text{Smaller}(a, y) \rightarrow \text{Large}(y))$

Proof: We will use general conditional proof to show our conclusion. Let c stand for an arbitrary block and assume for the sake of argument that $\text{Smaller}(a, c)$. We will try to show that $\text{Large}(c)$. From 1 we know that a is medium. But our assumption tells us that a is smaller than c , so c must be large. This is exactly what we were out to show to finish our general conditional proof, so we are done.

Formal Proofs

Getting Back Into the Swing of It

- Remember formal proofs?
 - Contains only wffs of FOL
 - Every non-premise or assumption line must cite a formal rule as its justification
- Example:

1	Tet(a) \rightarrow Smaller(a, c)	
2	Tet(a)	
3	Smaller(a, c)	\rightarrow Elim: 2, 1
4	Smaller(a, c) \vee LeftOf(a, b)	\vee Intro: 3
5	Tet(a) \rightarrow (Smaller(a, c) \vee LeftOf(a, b))	\rightarrow Intro: 2-4

Formal Proofs

Review

- Remember that for **every logical symbol** we have **two rules**:
 - 1 = **Elim**, = **Intro**
 - 2 \neg **Elim**, \neg **Intro**
 - 3 \perp **Elim**, \perp **Intro**
 - 4 \wedge **Elim**, \wedge **Intro**
 - 5 \vee **Elim**, \vee **Intro**
 - 6 \rightarrow **Elim**, \rightarrow **Intro**
 - 7 \leftrightarrow **Elim**, \leftrightarrow **Intro**
- Recall that each of these rules was the formal analog of an **informal** inference step or proof method
- The rules are justified by the **meaning** of each symbol
- Now, we'll learn the formal rules for \forall :
 \forall **Intro**, \forall **Elim**

 \forall **Elim**

Universal Elimination

Universal Elimination (Official Version, Informal Step)

If you have $\forall x S(x)$, you may infer $S(c)$, provided that 'c' refers to an object in the domain of discourse

 \forall **Elim**

$\forall x S(x)$ \vdots $\triangleright S(c)$

EXAMPLE:

1	$\forall x (\text{Tet}(x) \wedge \text{Small}(x))$	
2	$\text{Tet}(c) \wedge \text{Small}(c)$	\forall Elim: 1

 \forall **Intro**

A Preliminary Note

- There are two methods of **informally proving** universal claims:
 - 1 Universal Introduction
 - 2 General Conditional Proof
- We will consider two versions of \forall **Intro** corresponding to each of these methods

\forall Intro

Version 1

 \forall Intro Version 1 (Formal Method of Proof)
$$\begin{array}{|l} \boxed{c} \\ \vdots \\ P(c) \\ \hline \forall x P(x) \end{array}$$

c must not occur outside of the subproof where it is introduced

That is, c must be arbitrary

- Read the boxed ' c ' as *let c be an arbitrary member of the domain of discourse*

 \forall Intro, Version 1

An Example

$$\begin{array}{|l} 1 \quad \forall x (\text{Tet}(x) \wedge \text{Small}(x)) \\ \hline 2 \quad \boxed{c} \\ 3 \quad \text{Tet}(c) \wedge \text{Small}(c) \quad \forall \text{Elim: } 1 \\ 4 \quad \text{Tet}(c) \quad \wedge \text{Elim: } 3 \\ 5 \quad \forall x \text{Tet}(x) \quad \forall \text{Intro: } 2-4 \end{array}$$
 \forall Intro, Version 1

Another Example

$$\begin{array}{|l} 1 \quad \forall x \text{Tet}(x) \\ 2 \quad \forall x (\text{Tet}(x) \vee \text{Small}(x)) \end{array}$$

Proof: We will use universal introduction to prove the conclusion. Let c name an arbitrary block. From 1 $\text{Tet}(c)$ follows by universal elimination. But if *that* is true, so is $\text{Tet}(c) \vee \text{Small}(c)$ (by disjunction introduction). Since c was arbitrary, it follows that $\forall x (\text{Tet}(x) \vee \text{Small}(x))$.

$$\begin{array}{|l} 1 \quad \forall x \text{Tet}(x) \\ \hline 2 \quad \boxed{c} \\ 3 \quad \text{Tet}(c) \quad \forall \text{Elim: } 1 \\ 4 \quad \text{Tet}(c) \vee \text{Small}(c) \quad \vee \text{Intro: } 3 \\ 5 \quad \forall x (\text{Tet}(x) \vee \text{Small}(x)) \quad \forall \text{Intro: } 2-4 \end{array}$$
 \forall Intro, Version 1

Using Fitch

- Let's do **Exercise 13.2**
- But, we'll learn how to use Fitch to do it

$$\begin{array}{|l} 13.2 \quad \forall x (\text{Cube}(x) \leftrightarrow \text{Small}(x)) \\ \hline \forall x \text{Cube}(x) \\ \hline \forall x \text{Small}(x) \end{array}$$

\forall Intro

In Class Exercise

Give a *formal proof* that the following argument is valid

$$\begin{array}{l} \forall y \text{ Small}(y) \\ \forall x \text{ Cube}(x) \\ \hline \forall x (\text{Cube}(x) \wedge \text{Small}(x)) \end{array}$$

It may be helpful to first do an informal proof

Hint: Since you are trying to prove a universal claim, use the method of \forall **Intro**

 \forall Intro

The Rule, Version 2

General Conditional Proof (Informal Method of Proof)

- 1 Introduce a **new** name c to stand for a completely **arbitrary** member of the domain of discourse and assume $A(c)$
- 2 Prove $B(c)$
- 3 Conclude $\forall x (A(x) \rightarrow B(x))$

 \forall Intro Version 2 (Formal Method of Proof)

$$\begin{array}{l} \boxed{c} A(c) \\ \vdots \\ B(c) \\ \triangleright \forall x (A(x) \rightarrow B(x)) \end{array}$$

c must not occur outside of the subproof where it is introduced
That is, c must be arbitrary

- Read the boxed ' c ' as *let c be an arbitrary member of the domain of discourse*

 \forall Intro

General Conditional Proof

$$\begin{array}{l} 1 \quad \text{Medium}(a) \\ 2 \quad \forall y (\text{Smaller}(a, y) \rightarrow \text{Large}(y)) \end{array}$$

Proof: Let c be an arbitrary block & assume $\text{Smaller}(a, c)$. Since our premise says that a is medium, our assumption requires that $\text{Large}(c)$. Because c was arbitrary, the conclusion follows by General Conditional Proof.

$$\begin{array}{l} 1 \quad \text{Medium}(a) \\ 2 \quad \boxed{c} \text{Smaller}(a, c) \\ 3 \quad \text{Large}(c) \quad \text{Ana Con: 1,2} \\ 4 \quad \forall y (\text{Smaller}(a, y) \rightarrow \text{Large}(y)) \quad \forall \text{Intro: 2-3} \end{array}$$

 \forall Intro

An Example

$$\begin{array}{l} 1 \quad \forall x [\text{Tet}(x) \rightarrow \neg \text{Cube}(x)] \\ 2 \quad \forall x [\neg \text{Cube}(x) \rightarrow \text{Small}(x)] \\ 3 \quad \boxed{c} \text{Tet}(c) \\ 4 \quad \text{Tet}(c) \rightarrow \neg \text{Cube}(c) \quad \forall \text{Elim: 1} \\ 5 \quad \neg \text{Cube}(c) \quad \rightarrow \text{Elim: 3,4} \\ 6 \quad \neg \text{Cube}(c) \rightarrow \text{Small}(c) \quad \forall \text{Elim: 2} \\ 7 \quad \text{Small}(c) \quad \rightarrow \text{Elim: 5,6} \\ 8 \quad \forall x [\text{Tet}(x) \rightarrow \text{Small}(x)] \quad \forall \text{Intro: 3-7} \end{array}$$

Hint: working backwards is very helpful w/ \forall **Intro**

Another Example

In Fitch

Time permitting, in Fitch:

④ Exercise 13.7