

Informal Proofs with Quantifiers III

Mixed Proofs

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Outline

- 1 Review
- 2 Caution When Mixing Quantifiers
- 3 Proofs With Mixed Quantifiers

Announcements

04.16

- 1 I've updated your grades on Sakai, so if there are any outstanding issues, get in touch
- 2 There will be important in-class exercises next Tuesday so you are highly advised to attend

Two Inference Steps

In Review

Existential Introduction (Official Version)

$$\triangleright \frac{S(c)}{\exists x S(x)}$$

(When 'c' names an object in the domain of discourse)

Universal Elimination (Official Version)

$$\triangleright \frac{\forall x S(x)}{S(c)}$$

(Where 'c' refers to an object in the domain of discourse)

Existential Elimination

In Review

The Method of Existential Elimination

- 1 Given $\exists x S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c , and then **assume** $S(c)$
 - 2 However, c must be a **new name**, i.e. one not already in use in the context of your proof
- You are likely to need this method when you have an **existential premise**

Existential Elimination

An Example

Example Argument

1	$\forall x [\text{Tet}(x) \rightarrow \text{Small}(x)]$
2	$\exists x \text{Tet}(x)$
3	$\exists x \text{Small}(x)$

Proof:

- From 2 we know there is some block, call it d , such that $\text{Tet}(d)$ (Exist. Elim.)
- From 1 by Univ. Elim. we get $\text{Tet}(d) \rightarrow \text{Small}(d)$
- So we have $\text{Small}(d)$ by modus ponens
- By Exist. Intro. it follows that: $\exists x \text{Small}(x)$ ✓

Universal Introduction

The Official Formulation

Universal Introduction

To prove $\forall x S(x)$:

- 1 Introduce a **new** name c to stand for a completely **arbitrary** member of the domain of discourse
- 2 Prove $S(c)$
- 3 Conclude $\forall x S(x)$

- You will need to use this method whenever you are trying to prove a **universal claim**
- Since many of the universal claims we deal with are universal conditionals, we formulated a short-cut for these claims

Universal Introduction

An Example

1	$\forall y \text{LeftOf}(y, b)$
2	$\forall x [\text{LeftOf}(x, b) \rightarrow \text{Smaller}(x, a)]$
3	$\forall x \exists y \text{Smaller}(x, y)$

Proof: Let c be an arbitrary block. (*Goal:* show

$\exists y \text{Smaller}(c, y)$) From 1 we get $\text{LeftOf}(c, b)$, by Univ. Elim. From 2 we get $\text{LeftOf}(c, b) \rightarrow \text{Smaller}(c, a)$. So $\text{Smaller}(c, a)$ follows by modus ponens. By Exist. Intro. we get $\exists y \text{Smaller}(c, y)$. Since c was arbitrary, $\forall x \exists y \text{Smaller}(x, y)$ follows by Univ. Intro.

General Conditional Proof

In Review

General Conditional Proof

To prove $\forall x (A(x) \rightarrow B(x))$:

- 1 Introduce a **new** name c to stand for a completely **arbitrary** member of the domain of discourse
- 2 Assume $A(c)$
- 3 Prove $B(c)$
- 4 Conclude $\forall x (A(x) \rightarrow B(x))$

- Use this method to prove universal conditionals like $\forall x (\text{Cube}(x) \rightarrow \text{Small}(x))$

Mixing Quantifiers

In a Proof

- We will often want to use **both** Exist. Elim. and Univ. Intro. or Gen. Cond. Pf.
- Two of the important facts we've recently learned:
 - Using existential elimination requires the careful use of arbitrary names
 - Using universal introduction requires the careful use of arbitrary names
- An equally important consequence of these facts:

Important Fact about Mixing Quantifier Proof Methods

Using existential elimination and universal introduction **in the same proof** requires a doubly careful use of arbitrary names

Mixing Quantifiers

A Real Proof

- 1 $\exists y \forall x \text{Loves}(x, y)$
- 2 $\forall x \exists y \text{Loves}(x, y)$

Proof: We will show that $\exists y \text{Loves}(a, y)$, holds for an arbitrary a . Given the premise, at least one person is loved everyone. Let d be one of these lovely people: $\forall x \text{Loves}(x, d)$. (Exist. Elim.) By Univ. Elim. we get $\text{Loves}(a, d)$. By Exist. Intro. it follows that $\exists y \text{Loves}(a, y)$. Since a was arbitrary, it follows by Univ. Intro. that $\forall x \exists y \text{Loves}(x, y)$.

Mixing Quantifiers

The Opposite Inference is Invalid

- 1 $\forall x \exists y \text{Loves}(x, y)$
- 2 $\exists y \forall x \text{Loves}(x, y)$

- We know that this inference isn't valid
- Consider a world with two people:
 - Alice and Bob
 - Bob loves Alice
 - Alice loves Bob
 - But, Alice does not love herself
 - Neither does Bob
- The premise is true: everyone loves someone or other
- But the conclusion is false, no one is loved by everyone

Mixing Quantifiers

A Pseudo-Proof

- 1 $\forall x \exists y \text{ Loves}(x, y)$
- 2 $\exists y \forall x \text{ Loves}(x, y)$

Pseudo-Proof: The premise says that everyone likes someone or other. So let b be any boy, and g be a girl he loves. **Since b was chosen arbitrarily**, we may conclude by Univ. Intro. that $\forall x \text{ Loves}(x, g)$. By existential introduction, our conclusion follows.

- The crucial misstep in this proof was the claim that b was arbitrary
- By introducing g , a specific girl that b likes, **b ceases to be arbitrary**
 - Our proof then contains specific information about b : that he likes g , which is **not** true of everyone

Mixing Quantifiers

The Moral

Summary

- 1 **Universal Introduction:** To prove $\forall x S(x)$, chose a **new** constant c and prove $S(c)$, making sure that $S(c)$ does not contain any names introduced by Exist. Elim. after the introduction of c .
 - Above, g was introduced by Exist. Elim. after b
 - 2 The same goes for applications of General Conditional Proof
- If you do not follow this advice, you will be able to give ‘proofs’ of invalid arguments
 - But, if there’s proofs of invalid arguments, the whole idea of proof is bankrupt!

In-Class Exercise

Exercise 12.14

12.14 *This exercise contains a purported proof. If it is correct, say so. If it is incorrect, explain what goes wrong using the notions presented above.*

$$\exists x (x = x \rightarrow \neg \exists y x \neq y) \quad (\text{There is at most one object})$$

Purported proof: Toward a proof by contradiction, suppose $\neg \exists x (x = x \rightarrow \neg \exists y x \neq y)$. This is equivalent to $\forall x \neg (x = x \rightarrow \neg \exists y x \neq y)$, which is equivalent to $\forall x (x = x \wedge \exists y x \neq y)$. By Univ. Elim. we get $c = c \wedge \exists y c \neq y$. By Exist. Elim. we may assume $c \neq d$. But, since c was arbitrary, it follows that $\forall x x \neq d$. By Univ. Elim, we get $d \neq d$, which is a contradiction. Thus, the conclusion must be true.

Mixing Quantifiers in Proofs

Where We Are

- So far: an important lesson about how **not** to apply Univ. Intro. and Exist. Elim.
- We learned how to recognize this mistake
- But we also need to practice correctly mixing these two rules
- So let’s do some more informal proofs that require mixing the two rules

Mixing Quantifiers

An Example

$$\forall x \forall y [\text{Smaller}(x, y) \rightarrow \text{SameShape}(x, y)]$$

$$\forall x \exists y [\text{Adjoins}(x, y) \rightarrow \text{Smaller}(x, y)]$$

$$\forall x \exists y [\text{Adjoins}(x, y) \rightarrow \text{SameShape}(x, y)]$$

Proof: From premise 2 by Univ. Elim.
 $\exists y [\text{Adjoins}(c, y) \rightarrow \text{Smaller}(c, y)]$. By Exist. Elim. we may then assume $\text{Adjoins}(c, d) \rightarrow \text{Smaller}(c, d)$. From premise 1 by Univ. Elim. $\text{Smaller}(c, d) \rightarrow \text{SameShape}(c, d)$. By the transitivity of \rightarrow , we have $\text{Adjoins}(c, d) \rightarrow \text{SameShape}(c, d)$. Exist. Intro. then gives us $\exists y [\text{Adjoins}(c, y) \rightarrow \text{SameShape}(c, y)]$. Since c was arbitrary, the conclusion follows by Univ. Intro.

Mixing Quantifiers

Another Example

$$\exists y \forall x (\text{Smaller}(x, y) \vee \neg \text{Tet}(x))$$

$$\forall y \forall x (\text{Smaller}(y, x) \rightarrow \text{Cube}(y))$$

$$\forall z \forall y [(\text{LeftOf}(z, y) \wedge \neg \text{Tet}(z)) \rightarrow \text{Cube}(z)]$$

$$\forall x (\forall y \text{LeftOf}(x, y) \rightarrow \text{Cube}(x))$$

Proof: We will use general conditional proof, but first we apply Exist. Elim. to premise 1 and assume $\forall x (\text{Smaller}(x, a) \vee \neg \text{Tet}(x))$. Now we assume $\forall y \text{LeftOf}(c, y)$, with the goal of showing $\text{Cube}(c)$. This gives us $\text{LeftOf}(c, a)$ by Univ. Elim. By Univ. Elim. we also have $\text{Smaller}(c, a) \vee \neg \text{Tet}(c)$. Consider the second case. Premise 3 gives us $(\text{LeftOf}(c, a) \wedge \neg \text{Tet}(c)) \rightarrow \text{Cube}(c)$. Then $\text{Cube}(c)$ follows by modus ponens. In the second case, premise 2 gives us $(\text{Smaller}(c, a) \rightarrow \text{Cube}(c))$ by Univ. Elim. So we have $\text{Cube}(c)$ again by modus ponens. Thus, either way, we have $\text{Cube}(c)$, and since c was arbitrary the conclusion follows by general conditional proof.