Informal Proofs with Quantifiers II

Universal Proofs

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Announcements

04.14

1. HW10 is due now
2. The final exam is on May 13th from 8-11am
   - If you have a conflict, get in touch w/me ASAP

Outline

Two Inference Steps

Existential Introduction & Universal Elimination

Existential Introduction (Official Version)

$$\begin{array}{r|l} & \mathsf{S(c)} \\ \hline \triangleright & \exists x\, \mathsf{S(x)} \end{array}$$

(When ‘c’ names an object in the domain of discourse)

Universal Elimination (Official Version)

$$\begin{array}{r|l} & \forall x\, \mathsf{S(x)} \\ \hline \triangleright & \mathsf{S(c)} \end{array}$$

(Where ‘c’ refers to an object in the domain of discourse)

Two Inference Steps
A Simple Example

Example Argument

1	$\mathsf{Tet(a)} \rightarrow \forall x\, \mathsf{Small(x)}$
2	$\forall y\, \mathsf{Tet(y)}$
3	$\exists x\, \mathsf{Small(x)}$

Proof:

- From 2 by **universal elimination** we get Tet(a)
- From this and 1 we get by modus ponens $\forall x\, \mathsf{Small(x)}$

- Applying **universal elimination** to this, we get Small(a)
- By **existential introduction** it follows that: $\exists x\, \mathsf{Small(x)}$ ✓

Existential Elimination
Background

- Suppose you are given an existential premise and need to use it to prove a conclusion

  (1) *Something is a cube*
- Suppose the domain includes only two blocks *a* and *b*
- What can you infer from (1)?
  - *a* is a cube? No!
  - *b* is a cube? No!
- Here's an idea:
  - We can infer from (1) that there is some block, call it *Frank*, that is a cube
- Then we can continue on in our reasoning as if *Frank* was a real name, even though it's a dummy name (an ersatz)
- This *dummy name* method turns out to be **very** useful

Existential Elimination
In Review

The Method of Existential Elimination

1. Given $\exists x\, \mathsf{S(x)}$, you may give a dummy name to (one of) the object(s) satisfying S(x), say c, and then assume S(c)
2. However, c must be a new name, i.e. one not already in use in the context of your proof

- Remember, the whole idea of the dummy name is to remain agnostic about which object(s) satisfy S(x)
- In a proof with existential and universal premises:
  - Always apply existential elimination before applying universal elimination
  - This will save you space and possible confusion

Existential Elimination
An Example

Example Argument

1	$\forall z\, [\neg \mathsf{Tet(z)} \vee \mathsf{Cube(z)}]$
2	$\forall x\, [\neg \mathsf{Tet(x)} \rightarrow \mathsf{Small(x)}]$
3	$\exists x\, \neg \mathsf{Small(x)}$
4	$\exists x\, \mathsf{Cube(x)}$

Proof:

- First, we apply **existential elimination** to 3 $\neg$Small(a) (note 'a' is new)
- From 3 by **universal elimination** we get $\neg \mathsf{Tet(a)} \rightarrow \mathsf{Small(a)}$

- These two facts imply that $\neg$Tet(a) is false
- From 2 by **universal elimination** it follows that $\neg \mathsf{Tet(a)} \vee \mathsf{Cube(a)}$
- Since $\neg$Tet(a) is false, Cube(a) must be true
- By **existential introduction** it follows that $\exists x\, \mathsf{Cube(x)}$ ✓

Summary
The Steps and Methods So Far

Method of Existential Elimination

1. Given $\exists x\, S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c, and then assume $S(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

Existential Introduction (Official Version)

$$\begin{array}{l|l} & S(n) \\ \hline \triangleright & \exists x\, S(x) \end{array}$$

(When 'n' names an object in the domain of discourse)

Universal Elimination (Official Version)

$$\begin{array}{l|l} & \forall x\, S(x) \\ \hline \triangleright & S(c) \end{array}$$

(Where 'c' refers to an object in the domain of discourse)

What We've Done
Taking Stock

- We've learned two inference steps and one proof method for quantifiers:
  1. Universal Elimination, Existential Introduction
  2. The Method of Existential Elimination
- What's missing from this list?
  - Universal Introduction
- Universal introduction is a proof method and requires the appeal to dummy names familiar from existential elimination
- We'll start with some example inferences

Universal Introduction
Justifying a Universal

- Suppose you are looking at Tarski's World and there are 3 blocks: a, b and c
- Now suppose you are asked to prove the following universal claim:

  (2) $\forall x\, Tet(x)$
- How might you go about it?
  - Consider each object, and show that it satisfies $Tet(x)$
  - Cumulatively, this process will justify saying that (2) is true in this world
- Call this method the *check-each-object method*

Universal Introduction
The Need for a Better Method

- Consider the fact that:

  (2) $\forall x\, \neg[Cube(x) \wedge Tet(x)]$

  This is true of every world
- So, we should be able to prove (2) without considering particular objects from a particular world
- Further, we should be able to prove it even if there were infinitely many objects
- These two facts go against the check-each-object-method:
  - That method requires you to consider particular objects from a particular world
  - It also assumes that it is possible to finish checking every world
- Let's look at a more general method

Universal Introduction
An Example from Tarski's World

$\forall x \neg[\text{Cube}(x) \wedge \text{Tet}(x)]$

Proof: Let c be an arbitrary block. If we assume $\text{Cube}(c) \wedge \text{Tet}(c)$, then we immediately have a contradiction, since c cannot be both a cube and a tetrahedron. So it must be true that $\neg[\text{Cube}(c) \wedge \text{Tet}(c)]$ But since c was an **arbitrarily chosen** block, it must be that $\forall x \neg[\text{Cube}(x) \wedge \text{Tet}(x)]$.

- The key in this proof is the use of a dummy name to talk about an arbitrary block

Universal Introduction
An Example from the Real World

Anyone who passes Phil 201 with an A is smart

Every math major has passed Phil 201 with an A

Every math major has been smart

Proof: Let 'Jessica' refer to any one of the math majors. By the second premise, Jessica must have passed Phil 201 with an A (universal elimination). Then by the first premise, Jessica must have been smart. But since Jessica was an **arbitrarily** chosen math major, it follows that every math major was smart.

- The key in this proof is the use of a dummy name to talk about an arbitrary math major

Universal Introduction
The Important Features of Our Proof

- Notice in our proofs we didn't need to consider a particular set of blocks or math majors
- Our proof method was perfectly general: it works regardless of which set of entities you apply it to
- This generality was achieved by introducing a new name to talk about an arbitrary entity
- Since that entity was selected arbitrarily, when we inferred something about that entity, we were entitled to conclude something about every object
- This is the basic idea behind Universal Introduction

Universal Introduction
The Official Formulation

Universal Introduction

To prove $\forall x\, S(x)$:

1. Introduce a new name c to stand for a completely arbitrary member of the domain of discourse
2. Prove $S(c)$
3. Conclude $\forall x\, S(x)$

- Since c is arbitrary, showing $S(c)$ amounts to showing $\forall x\, S(x)$
- c's being arbitrary prevents one from assuming that any properties specific to one object are used in the course of the proof

Universal Introduction
Another Example

Example Argument

1	$\forall x\, \mathsf{Tet}(x)$
2	$\forall x\, \mathsf{Medium}(x)$
3	$\forall x\, (\mathsf{Tet}(x) \land \mathsf{Medium}(x))$

Proof:

- Let 'c' be an arbitrary block
- From 1 $\mathsf{Tet}(c)$ follows by universal elimination
- Applying universal elimination to 2 gives us $\mathsf{Medium}(c)$
- So we have $\mathsf{Tet}(c) \land \mathsf{Medium}(c)$
- But c was arbitrary, so it follows that $\forall x\, (\mathsf{Tet}(x) \land \mathsf{Medium}(x))$ ✓

Universal Introduction
In Class Exercise

Give an informal proof for:

1	$\forall y\, \mathsf{LeftOf}(y, b)$
2	$\forall x\, [\mathsf{LeftOf}(x, b) \to \mathsf{SameSize}(x, a)]$
3	$\forall x\, \exists y\, \mathsf{SameSize}(x, y)$

Hint: use universal introduction. Premise 2 says *Every block left of **b** is smaller than **a***. The conclusion says that *Every block is smaller than some block or other.*

General Conditional Proof
How to Prove a Universal Conditional

- In practice, we are usually concerned with proving universal claims of these forms:
  - *Every A is B*
  - *All A are B*, etc.
- As we all know, these are translated in FOL as:

$$\forall x\, (A(x) \to B(x))$$

- To prove this using universal introduction you would prove, for an arbitrary c:

$$A(c) \to B(c)$$

- This would be achieved using conditional proof:
  - Assume $A(c)$ and show $B(c)$

Conditional Proof
Review of Conditional Proof

The Method of Conditional Proof

To prove $P \to Q$, temporarily assume P. If you can show Q with this additional assumption, you can infer $P \to Q$

Truth Table for $\to$

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

- The only way for $P \to Q$ to be F is for P to be true and Q be F
- So, if you can show that when P is T Q is also T, you've shown that $P \to Q$ cannot be false, i.e. that it is true!

Conditional Proof

Review of Conditional Proof: An Example

Let's use conditional proof and modus ponens to give a proof of:

ARGUMENT 1

$$\begin{array}{|l} \text{Tet}(a) \rightarrow \text{Tet}(b) \\ \text{Tet}(b) \rightarrow \text{Tet}(c) \\ \hline \text{Tet}(a) \rightarrow \text{Tet}(c) \end{array}$$

Our goal is a conditional, so we use conditional proof.

Proof: Suppose Tet(a). Then by premise 1 Tet(b) follows by modus ponens. But then we may now again use modus ponens and premise 2 to infer Tet(c). This is the consequent of our goal, so we have successfully completed our conditional proof.

General Conditional Proof

Universal Instantiation Plus Conditional Proof

- Proofs will often involve using conditional proof & universal introduction together
- So, let's introduce a short-cut & name for it

General Conditional Proof

To prove $\forall x\, (A(x) \rightarrow B(x))$:

1. Introduce a new name c to stand for a completely arbitrary member of the domain of discourse
2. Assume A(c)
3. Prove B(c)
4. Conclude $\forall x\, (A(x) \rightarrow B(x))$

- This is equivalent to using universal introduction along with conditional proof

General Conditional Proof

An Example

Example Argument

$$\begin{array}{|l} \forall x\, [(\text{Small}(x) \rightarrow \neg\text{Tet}(x)) \\ \forall x\, [\neg\text{Tet}(x) \rightarrow \text{Cube}(x)] \\ \hline \forall x\, [\text{Small}(x) \rightarrow \text{Cube}(x)] \end{array}$$

Proof:

- Let a name an arbitrary block
- Suppose Small(a)
  (Goal: Show Cube(a))
- From premise 1: $\text{Small}(a) \rightarrow \neg\text{Tet}(a)$
  - By modus ponens, we get ¬Tet(a
  - Premise 2 gives us $\neg\text{Tet}(a) \rightarrow \text{Cube}(a)$, so by modus ponens we have Cube(a), (our goal)
  - Since a was arbitrary, it follows that $\forall x\, [\text{Small}(x) \rightarrow \text{Cube}(x)]$ ✓

General Conditional Proof

Another Example

Example Argument

$$\begin{array}{|l} \forall x\, [(\text{Cube}(x) \lor \text{Large}(x)) \\ \quad \lor (\text{Medium}(x) \land \text{Tet}(x))] \\ \forall x\, [\text{Tet}(x) \rightarrow \neg\text{Smaller}(x, c)] \\ \hline \forall x\, [\text{Medium}(x) \rightarrow \neg\text{Smaller}(x, c)] \end{array}$$

Proof:

- Let a name an arbitrary block
- Suppose Medium(a)
  (Goal: Show ¬Smaller(a, c))
- From premise 1: $(\text{Cube}(a) \land \text{Large}(a)) \lor (\text{Medium}(a) \land \text{Tet}(a))$
  - Since Medium(a), the first disjunct must be false, and $\text{Medium}(a) \land \text{Tet}(a)$ must be true
  - Premise 2 gives us $\text{Tet}(a) \rightarrow \neg\text{Smaller}(a, c)$, so by modus ponens we have ¬Smaller(a, c), (our goal)
  - Since a was arbitrary, it follows that $\forall x\, [\text{Medium}(x) \rightarrow \neg\text{Smaller}(x, c)]$ ✓

General Conditional Proof

In Class Exercise

Give an informal proof for:

1	$\forall y\,[\exists x\,\mathsf{Tet}(x) \rightarrow \mathsf{LeftOf}(y, b)]$
2	$\forall x\,[\mathsf{LeftOf}(x, b) \rightarrow \mathsf{Smaller}(x, a)]$
3	$\forall x\,[\mathsf{Tet}(x) \rightarrow \mathsf{Smaller}(x, a)]$

Hint: use the method of general conditional proof, along with universal elimination, existential introduction and modus ponens.

Universal Proof

Summary

Summary

1. To prove a universally quantified claim, use Universal Introduction
   - E.g. to prove $\forall x\,\mathsf{Tet}(x)$, use Univ. Intro.
2. When proving a universal conditional, you may use General Conditional Proof
   - This is just Univ. Intro. together with Conditional Proof
3. These are both proof methods

- Next class, we will learn how to mix Univ. Intro. with the method of Existential Elimination