Informal Proofs with Quantifiers I

Inference Steps and Existential Instantiation

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Outline

Shifting Gears

Proof

- We have learned about what quantifiers mean
- Now it is time to think about how quantificational sentences should be used in proofs
- We have not done any proofs for a while, so let's remind ourselves of what they are all about
- Proofs are step-by-step demonstrations of a conclusion from some premises
  - Each step is justified by the meanings of the terms involved
- Proofs can be formal or informal

Inference Steps

What They Are

- An inference step is a simple transition from one claim C_1 to another C_2
- Valid inference steps are ones where the truth of C_1 guarantees the truth of C_2
- So far, we have not learned which steps involving quantifiers are valid
- In this section of the lecture we are going to learn two important inference steps for quantifiers

Universal Elimination
An Example

- Suppose you are really convinced of this generalization:

  (1) *Everyone has DNA*

- Now consider a random person: Michael Jackson
- Given (1), what can we infer about MJ?

  (2) *MJ has DNA*

- Why?
- (2) logically follows from (1)!
  - If (1) is true it is impossible for (2) to be false!

Universal Elimination
The Inference Pattern at Work

(1) *Everyone is has DNA*

(2) *Michael Jackson has DNA*

- (2) is a logical consequence of (1)
- But this is just one example of more general valid inference pattern:

Universal Elimination (Unofficial Version)

	Everything is an F
▷	c is an F

(Where 'c' is a name for an actually existing object)

Universal Elimination
The Official Version

- Our unofficial version of universal elimination gets the basic idea right
- But, it's not quite as general as it should be
- To make it more general, it is helpful to write it in terms of FOL:

Universal Elimination (Official Version)

From $\forall x\, S(x)$ you may infer $S(c)$, as long as 'c' refers to an object in the domain of discourse.

- This is an informal inference step

Universal Elimination
Another Example

Universal Elimination (Official Version)

From $\forall x\, S(x)$ you may infer $S(c)$, as long as 'c' refers to an object in the domain of discourse.

- Suppose you are given:

  (3) $\forall x\, (\mathsf{Cube}(x) \lor \mathsf{Small}(x))$

  And you also know that a, b name objects in the domain of discourse

- Then you can infer, by universal elimination:

  (4) $\mathsf{Cube}(a) \lor \mathsf{Small}(a)$

  (5) $\mathsf{Cube}(b) \lor \mathsf{Small}(b)$

Existential Introduction
An Example

(6) *George Bush greeted the Pope*

- So:

  (7) *Someone met the pope*

(8) *Everyone loves Plato*

- So:

  (9) *Everyone loves someone*

Existential Introduction (Unofficial Version)

| c is an F
|---
▷ | Something is an F

Existential Introduction
The Official Version

- Again, this informal rule is clearer and more general when stated using FOL:

Existential Introduction (Official Version)

From $\mathsf{S(c)}$ you may infer $\exists \mathsf{x}\, \mathsf{S(x)}$, as long as 'c' refers to an object in the domain of discourse.

- Example:

  (10) $\mathsf{Tet(a)} \lor \neg \mathsf{SameSize(a, c)}$

  Therefore, by existential introduction:

  (11) $\exists \mathsf{x}\, (\mathsf{Tet(x)} \lor \neg \mathsf{SameSize(x, c)})$

Existential Introduction

Existential Introduction (Official Version)

From $\mathsf{S(c)}$ you may infer $\exists \mathsf{x}\, \mathsf{S(x)}$, as long as 'c' refers to an object in the domain of discourse.

- So from:

  (12) *Santa Claus does not exist*

  Does it follow by existential introduction that:

  (13) There exists an x that does not exist

- No! *Santa Claus* does not name an object in the domain of discourse

An Example Proof
Putting Together our Two Inference Steps

Example Argument

1 | $\forall \mathsf{x}\, [\mathsf{Tet(x)} \rightarrow \mathsf{Small(x)}]$
2 | $\mathsf{Tet(a)}$
3 | $\exists \mathsf{x}\, [\mathsf{Tet(x)} \land \mathsf{Small(x)}]$

Proof:

- From 1 by universal elimination we get $\mathsf{Tet(a)} \rightarrow \mathsf{Small(a)}$
- From this and 2 we get by modus ponens $\mathsf{Small(a)}$

- So we have $\mathsf{Tet(a)} \land \mathsf{Small(a)}$
- By existential introduction it follows that: $\exists \mathsf{x}\, [\mathsf{Tet(x)} \land \mathsf{Small(x)}]$ ✓

Summary
Two Inference Steps

Summary

- For the quantifiers ∀ and ∃ there are two informal valid inference steps:
  1. **Universal Elimination**: from $\forall x\, S(x)$ you can infer $S(c)$, as long as c names an object in the domain
  2. **Existential Introduction**: from $S(c)$ you can infer $\exists x\, S(x)$, as long as c names an object in the domain
- There are two other, more involved, methods of proof for the quantifiers

Today, we'll learn one of them: existential elimination

Existential Elimination
Background

- Suppose you are given an existential premise and need to use it to prove a conclusion

  (14) *Something is either a cube or not small*
- Suppose the domain includes only two blocks *a* and *b*
- What can you infer from (14)?
  - *a* is a cube or not small? No!
  - *b* is a cube or not small? No!
- Here's an idea:
  - We can infer from (14) that there is some block, call it *Frank*, that is either a cube or not small
- Then we can continue on in our reasoning as if *Frank* was a real name, even though it's a dummy name (an ersatz)
- This dummy name method turns out to be **very** useful

Existential Elimination
An Example

Example Argument

1	$\forall x\, [Tet(x) \rightarrow Small(x)]$
2	$\exists x\, Tet(x)$
3	$\exists x\, Small(x)$

Proof:

- We need to use 2; let's try the dummy name method
- From 2 we know there is some block, call it d, such that $Tet(d)$
- From 1 by universal elimination we get $Tet(d) \rightarrow Small(d)$
- So we have $Small(d)$ by modus ponens
- By existential introduction it follows that: $\exists x\, Small(x)$ ✓

Existential Elimination
An Observation

Example Argument

1	$\forall x\, [Tet(x) \rightarrow Small(x)]$
2	$\exists x\, Tet(x)$
3	$\exists x\, Small(x)$

Observation:

- In our proof of this argument we introduced our dummy name and **then** used universal elimination
- Could we have done it in the opposite order?
- Suppose from 1 by universal elimination we get $Tet(d) \rightarrow Small(d)$
- Can we then use our dummy name method as say: *let d be whatever is a tet by 2*
- No! The essence of the dummy name method is to introduce a new name, but d is already in use here
- Notice if we pick a different dummy name, say e, it will get us nowhere, unless we then use universal elimination all over again to get $Tet(e) \rightarrow Small(e)$

Existential Elimination
Our Observation

- To summarize:
  - Always apply universal elimination after invoking the dummy name method
- While we're at it, let's come up with a better name and description of the 'dummy name method':

Method of Existential Elimination

1. Given $\exists x\, S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c, and then assume $S(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

Existential Elimination
Official Formulation

Method of Existential Elimination

1. Given $\exists x\, S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c, and then assume $S(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

- Remember, the whole idea of the dummy name is to remain agnostic about what object(s) satisfy $S(x)$
- Using an old name would violate this agnosticism
  - Old names are real names
  - And real names name particular objects; dummy names don't

Summary
Existential Elimination

Summary

- Existential elimination is a method of proof
- It gives you a tool for using $\exists x\, S(x)$ in further reasoning:
  - It allows you to talk about the thing that satisfies $S(x)$ by giving it a temporary name
  - But keep in mind that this must be a new name since $\exists x\, S(x)$ does not allow you to infer that some particular thing satisfies $S(x)$
- When doing a proof with universal and existential premises, always use existential elimination before universal elimination

In Class Exercise

Give an informal proof that the following argument is valid:

1	$\forall x\,[\mathsf{Tet}(x) \vee \neg \mathsf{Small}(x)]$
2	$\forall y\,[\mathsf{Tet}(y) \rightarrow \mathsf{LeftOf}(a, y)]$
3	$\exists x\, \mathsf{Small}(x)$
4	$\exists x\, \mathsf{LeftOf}(a, x)$

You may use any of the proof methods or inference steps discussed so far in this class

Summary

The Steps and Methods from Today

Method of Existential Elimination

1. Given $\exists x\, S(x)$, you may give a dummy name to (one of) the object(s) satisfying $S(x)$, say c, and then assume $S(c)$
2. However, c must be a new name, i.e. one not already in use in the context of your proof

Existential Introduction (Official Version)

From $S(c)$ you may infer $\exists x\, S(x)$, as long as 'c' refers to an object in the domain of discourse.

Universal Elimination (Official Version)

From $\forall x\, S(x)$ you may infer $S(c)$, as long as 'c' refers to an object in the domain of discourse.