Multiple & Mixed Quantifiers

Understanding Quantification

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04.02.09

Announcements

04.02

1. HW9 is due next Tuesday

Outline

Quantification

What We've Done

- So far, we've learned what $\forall$ and $\exists$ mean
  - Recall the **semantics** and **game rules**
  - Both were based on the concept of satisfaction
- We've also learned how to use $\forall$ and $\exists$ to translate some basic quantificational English sentences into FOL
  - Remember the four Aristotelian Forms
- Most recently, we learned about two logical concepts related to $\forall$ and $\exists$
  - FO Validity
  - FO Consequence
  - We test for both by using the replacement method

Quantification

We are Just Getting Started

- This is a good start, but there is a lot more to understanding the logic of quantifiers
- Today we are going to think about what sentences containing multiple quantifiers mean
- As well as how to translate them into FOL
- Recall that we've only looked at sentences containing one quantifier:
  - *All mops are smelly*
  - *Some ninjas are not sociable*
- But what happens when there are two, three or four?

Quantification

Multiple Quantifiers

- Recall what old Abe said:

  > *You may fool all of the people some of the time; you can even fool some of the people all of the time; but you can't fool all of the people all of the time*

- Count the quantifiers: 6!
- The point is:
  - We often communicate logically interesting things with several quantifiers
- So, as students of logic, we need to see how far what we've learned about sentences containing one quantifier takes us in understanding sentences with several

Multiple Existentials

A Simple Example

- We will begin by considering sentences with multiple occurrences of one quantifier

  (1) *Some cube is left of some tetrahedron*
- How should we represent (1) in FOL?
- We have many options
- Let's consider and compare them

Multiple Existentials

Translating our Simple Example

(1) *Some cube is left of some tetrahedron*

- Two (of the many) correct translations:

  (1a) $\exists x\, \exists y\, [\mathsf{Cube}(x) \wedge \mathsf{Tet}(y) \wedge \mathsf{LeftOf}(x, y)]$
  - There are objects x and y such that: x is a cube, y is a tetrahedron and x is left of y

  (1b) $\exists x\, [\mathsf{Cube}(x) \wedge \exists y\, (\mathsf{Tet}(y) \wedge \mathsf{LeftOf}(x, y))]$
  - There is an object x such that x is a cube and there exists an object y such that y is a tetrahedron and x is left of y
- (1a) stacks all of the quantifiers at the beginning
  - This makes it easier to paraphrase
  - But less like the English sentence (1)

Multiple Existentials

Multiplicity of Translations

- In addition to:

  (1a) $\exists x\, \exists y\, [\mathsf{Cube}(x) \wedge \mathsf{Tet}(y) \wedge \mathsf{LeftOf}(x, y)]$

  (1b) $\exists x\, [\mathsf{Cube}(x) \wedge \exists y\, (\mathsf{Tet}(y) \wedge \mathsf{LeftOf}(x, y))]$

  - We can put the quantifiers in reverse order:

    (1c) $\exists y\, \exists x\, [\mathsf{Cube}(x) \wedge \mathsf{Tet}(y) \wedge \mathsf{LeftOf}(x, y)]$

    (1d) $\exists y\, [\mathsf{Tet}(y) \wedge \exists x\, (\mathsf{Cube}(x) \wedge \mathsf{LeftOf}(x, y))]$

  - Or put the predicates in a different order:

    (1e) $\exists x\, \exists y\, [\mathsf{Tet}(y) \wedge \mathsf{Cube}(x) \wedge \mathsf{LeftOf}(x, y)]$

    (1f) $\exists x\, [\mathsf{Cube}(x) \wedge \exists y\, (\mathsf{LeftOf}(x, y) \wedge \mathsf{Tet}(y))]$

- Let's look at these in Tarski's World to see that they are equivalent (`Equivalences.sen/.wld`)

Translation Convention

A Helpful Note

Translation Conventions (Stylistic Advice)

1. All quantifiers are stacked up 'out in front'
2. The first quantifier in the English sentence is written first and binds x, the second goes second and binds y, the third goes third and binds z and so on
3. As much as possible, predicates are listed in the order that they appear:
   - We translate *some cube is left of some tetrahedron*:

     (1a) $\exists x\, \exists y\, [\mathsf{Cube}(x) \wedge \mathsf{Tet}(y) \wedge \mathsf{LeftOf}(x, y)]$
   - Rather than

     (1e) $\exists x\, \exists y\, [\mathsf{Tet}(y) \wedge \mathsf{Cube}(x) \wedge \mathsf{LeftOf}(x, y)]$
   - $\mathsf{Left}(x, y)$ **has** to go last, but $\mathsf{Cube}(x)$ goes before $\mathsf{Tet}(y)$, since it can

Translation

Comments on Our Convention

- In general, there are very many different but equally correct ways of translating quantified sentences
  - Especially in sentences with multiple quantifiers
  - By equally correct we mean FO Equivalent
- The conventions on the previous slide are purely **aesthetic**
- When all of a formula's quantifiers are stacked up in front of the formula, it is said to be in prenex form
- Everything we've said so far also holds for sentences containing multiple universal quantifiers

Multiple Universals

(2) *Every tetrahedron is larger than every cube*

- Given our conventions, the natural translation is:

  (2a) $\forall x\, \forall y\, [(\mathsf{Tet}(x) \wedge \mathsf{Cube}(y)) \rightarrow \mathsf{Larger}(x, y)]$

  - For every block x and every block y, if x is a tetrahedron and y is a cube then x is larger than y
- But this is equivalent to (among others):

  (2b) $\forall x\, [\mathsf{Tet}(x) \rightarrow \forall y\, (\mathsf{Cube}(y) \rightarrow \mathsf{Larger}(x, y))]$
- Let's look at Tarski's World (`Equivalences.sen/.wld`)

Multiple Quantifiers

An Important Fact

Fact 1 (Multiplied Quantifiers)

When you have multiple occurrences of a single quantifier, order does not matter:

1. $\exists x \exists y\, \mathsf{P}(x, y) \Leftrightarrow \exists y \exists x\, \mathsf{P}(x, y)$
2. $\forall x \forall y\, \mathsf{P}(x, y) \Leftrightarrow \forall y \forall x\, \mathsf{P}(x, y)$

A Tricky Fact

Resisting the Temptation...

- It is tempting to paraphrase:

(3) $\forall x \forall y\, [(\mathsf{Small}(x) \land \mathsf{Cube}(y)) \rightarrow \mathsf{RightOf}(x, y)]$

As:

(4) For every block x and every **other** block y, if x is small and y is a cube then x is right of y

- But (4) is not what (3) means
- (4) is really a paraphrase of:

(5) $\forall x \forall y\, [(x \neq y \land \mathsf{Small}(x) \land \mathsf{Cube}(y)) \rightarrow \mathsf{RightOf}(x, y)]$

- (3) and (5) are not equivalent
- Let's see this in Tarski's World (`Identity.sen`, `Identity.wld`)

The Tricky Fact

The Moral of the Story

The Tricky Fact

1. When evaluating sentences with multiple quantifiers, don't fall into the trap of thinking that distinct variables range over distinct objects
2. In fact, $\forall x \forall y\, \mathsf{P}(x, y)$ logically entails $\forall x\, \mathsf{P}(x, x)$, so the variables can't be assumed to range over distinct variables. (The same goes for $\exists$)

Mixing Quantifiers

Doing Things Differently

- In addition to repeating the same quantifier, you can mix quantifiers:

(6) *Everyone loves someone or other*

(7) *There is someone that everyone loves*

- Both (6) and (7) mix a universal and an existential
- But, they do it differently:
  - (6) is a *Universal Existential*
  - (7) is an *Existential Universal*
- Accordingly, we translate (6) and (7) differently:

(6′) $\forall x \exists y\, (\mathsf{Love}(x, y))$

(7′) $\exists y \forall x\, (\mathsf{Love}(x, y))$

Mixing Quantifiers
The Difference in Meaning is Big

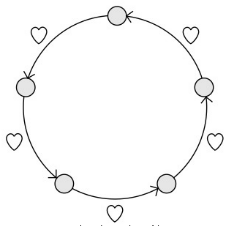
(6) *Everyone loves someone or other*

(6′) $\forall x \, \exists y \, (\mathsf{Love}(x, y))$

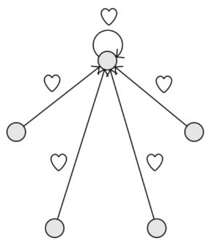
(7) *There is someone that everyone loves*

(7′) $\exists y \, \forall x \, (\mathsf{Love}(x, y))$

- (6)/(6′) and (7)/(7′) describe different situations:



(6),(6′)



(7),(7′)

Mixing Quantifiers
Entailment Relations

(6) *Everyone loves someone or other*

(6′) $\forall x \, \exists y \, (\mathsf{Love}(x, y))$

(7) *There is someone that everyone loves*

(7′) $\exists y \, \forall x \, (\mathsf{Love}(x, y))$

Fact

(7) entails (6). By (7) there's some person, call him/her Pat, that everyone loves. It follows that everyone loves someone (or other), namely Pat!

Fact

(6) does not entail (7). Everyone could love a different person. Then (6) is true but (7) is not

Mixing Quantifiers
The Important Difference

- What examples (6) and (7) show is that when you mix quantifiers order does matter!
- This is very different from multiple occurrences of a single quantifier:
  - Recall that when you have multiple quantifiers, order does not matter
- To solidify the difference between *existential-universal* and *universal-existential* let's look at some examples in Tarski's World (`MQ World.wld`, `MQ World 2.wld`, `MQ Sentences.sen`)

Summary
Two Facts

Fact 1 (Multiplied Quantifiers)

When you have multiple occurrences of a single quantifier, order does not matter:

1. $\exists x \, \exists y \, \mathsf{P}(x, y) \Leftrightarrow \exists y \, \exists x \, \mathsf{P}(x, y)$
2. $\forall x \, \forall y \, \mathsf{P}(x, y) \Leftrightarrow \forall y \, \forall x \, \mathsf{P}(x, y)$

Fact 2 (Mixed Quantifiers)

When you have multiple occurrences of different quantifiers, order does matter:

- $\forall x \, \exists y \, \mathsf{P}(x, y) \not\Leftrightarrow \exists y \, \forall x \, \mathsf{P}(x, y)$

Exercise

Mixed Quantifiers in Tarski's World

Let's work through **exercise 11.11** together