The Logic of Quantifiers

Logical Truth & Consequence in Full FOL

William Starr

03.31.09

Announcements

03.31

1. The Midterm has been returned
   - If you haven't gotten yours back, see me after class

Outline

1. Introduction
2. FO Validity
3. FO Consequence

Overview

The Big Picture

- Now that we've added $\forall$ and $\exists$, we have introduced every connective of FOL:

$$\forall \quad \exists \quad \leftrightarrow \quad \rightarrow \quad \vee \quad \wedge \quad \neg \quad =$$

- For six of these symbols we've studied:
  1. It's semantics: truth-tables, satisfaction, game rules
  2. How to translate English sentences using it
  3. It's role in logic: which sentences containing it are logical truths and which arguments containing it are valid
  4. It's role in proofs: which inference steps and methods of proof it supports and how these can be formalized
- For $\forall$ and $\exists$, we've only done the first two
- Today, we'll get started on the third!

Overview

Today

- So today we'll be interested in two questions:
  - Which sentences containing quantifiers are logical truths?
  - Which arguments containing quantifiers are valid?
- We'll start by reviewing our past discussion of logical truths and logical consequence

The Logical Concepts

Logical Truth & Logical Consequence

Logical Truth

A is a logical truth iff it is **impossible** for A to be false given the meaning of the logical vocabulary it contains

Logical Consequence

C is a logical consequence of $P_1, \ldots, P_n$ iff it is **impossible** for $P_1, \ldots, P_n$ to be true while C is false

- Both of these concepts are at the very heart of logic
  - But, they are annoyingly vague and imprecise
  - What exactly is meant by *impossible*?
- In the first half of the class we explored one method for making logical possibility precise: truth tables

Truth Tables

Their Spoils

- Truth tables allowed us to define the following concepts:

  Tautology

  A is a tautology iff every row the truth table assigns T to A

  Tautological Consequence

  C is a tautological consequence of $P_1, \ldots, P_n$ iff every row of their joint truth table which assigns T to $P_1, \ldots, P_n$ also assigns T to C

Truth Tables

Their Drawbacks

- These definitions are a step towards better understanding logical truth and consequence:
  - Every tautology is an (intuitive) logical truth
  - Every tautological consequence is an (intuitive) logical consequence
- But the step is not complete:
  - Some logical truths are not tautologies
  - Some logical consequences are not tautological consequences
- The difficulty was that the notion of logical possibility used in truth tables was not discerning enough

Truth Tables

Not Discerning Enough

- Recall the procedure for building a truth-table:

1. Build ref. col's
2. Fill ref. col's
3. Fill col's under connectives

Truth Table

$a = a$	$b = b$	$a = a \land b = b$
T	T	T
T	F	F
F	T	F
F	F	F

- This table shows that $a = a \land b = b$ is not a tautology: there are some F's in the main column
- But it is, intuitively, a logical truth

Truth Tables

Not Discerning Enough

1. Build ref. col's
2. Fill ref. col's
3. Fill col's under connectives

Truth Table

$a = a$	$b = b$	$a = a \land b = b$
T	T	T
T	F	F
F	T	F
F	F	F

- The problem is caused by the fact that in building truth tables, possibilities are included which are not genuine logical possibilities
  - It is not logically possible for $a = a$ or $b = b$ to be F!

Discussion

Truth Tables & Logical Possibility

- The same deficiency causes there to be logical consequences which are not tautological consequences
  - Example: $a = c$ is a logical but not a tautological consequence of $a = b \land b = c$
- Why not just leave rows out if they aren't genuine logical possibilities?
- This robs truth tables of their purpose:
  - They were supposed to be a precise way of analyzing logical possibility
  - If we can just appeal to our intuitions about logical possibility in building the columns, our analysis gets us nowhere
- So, we want to develop a better analysis of logical possibility

Tying In Quantification

We Need That Better Analysis Even More

- In case you weren't already convinced that truth tables left something to be desired, think about how few of the quantificational logical truths are tautologies
  - $\forall x\,(\mathsf{Cube}(x) \to \mathsf{Cube}(x))$ (Not a Tautology)
  - $\forall x\,(\mathsf{Cube}(x) \lor \neg\mathsf{Cube}(x))$ (Not a Tautology)
  - $\exists x\,(x = x)$ (Not a Tautology)
- Although some logical truths with quantifiers are tautologies:
  - $\forall x\,\mathsf{Cube}(x) \lor \neg\forall x\,\mathsf{Cube}(x)$ (Tautology)
  - $\neg(\exists x\,\mathsf{Cube}(x) \land \neg\exists x\,\mathsf{Cube}(x))$ (Tautology)

FO Validity
A Small Step

Logical Truth

A is a logical truth iff it is **impossible** for A to be false given the meaning of the logical vocabulary it contains

- We are only interested in $\forall, \exists, \leftrightarrow, \rightarrow, \vee, \wedge, \neg$ and $=$, so we are interested in a more limited concept

First-Order Validity (FO Validity)

A sentence A is a first-order validity just in case it is impossible for A to be false, given the meanings of $\forall, \exists, \leftrightarrow, \rightarrow, \vee, \wedge, \neg$ and $=$

- Better named *First-Order Logical Truth*

FO Validity
An Idea

- We need to be more clear about the notion of logical possibility used to define FO validity
- Here's the insight we'll build on
- The FO validities are sentences which are true purely in virtue of the meaning of $\forall, \exists, \leftrightarrow, \rightarrow, \vee, \wedge, \neg$ and $=$
- If their truth derives solely from the logical symbols, then you should be able vary the meaning of any of its predicates (other than $=$) and names and still get a true sentence
- Any variation of the meaning of the non-logical symbols is a logical possibility

An Example
Use a Non-Sense Predicate

(1) $\forall x\,(\mathsf{Cube}(x) \rightarrow \mathsf{Cube}(x))$

- It sounds true even with a non-sense predicate:
  (2) $\forall x\,(\mathsf{Blornk}(x) \rightarrow \mathsf{Blornk}(x))$
  (3) *All blornks are blornks*
- There's no interpretation of 'Blornk' according to which (2) isn't true
- So (1) remains true no matter how we interpret its non-logical symbols
- So (1) must be a FO validity

Another Example
Use a Non-Sense Predicate

(4) $\forall x\,\mathsf{Rich}(x) \rightarrow \mathsf{Rich}(\mathsf{mc.hammer})$

- This sounds true even with non-sense predicates and names:
  (5) $\forall x\,\mathsf{Rorg}(x) \rightarrow \mathsf{Rorg}(\mathsf{dude})$
  (6) *If everything is a rorg, then dude is a rorg*
- So (4) must be a FO validity

Yet Another Example

Use a Non-Sense Predicate

(7) $\neg\exists x\,\mathsf{LeftOf}(x, x)$

- Replace meaningful predicate with meaningless one:
  (8) $\neg\exists x\,\mathsf{Glirs}(x, x)$
- Is this obviously true?
- No, it would depend on whether or not something can glir itself
- This is a not a fact about the meaning of logical symbols, so this is not a FO validity

Counterexamples

What They Are

(7) $\neg\exists x\,\mathsf{LeftOf}(x, x)$

- We saw that, intuitively, (7) is not a logical truth
- But we want to have a more precise way of showing this
- Here's our new method:
  1. Replace predicates and names with non-sense names when checking for FO validity
  2. Then consider whether or not there is any reinterpretations of the formula that falsify it
  3. If there are, specify such an interpretation
     - This specification is called a *counterexample*
  4. If there is no such specification, then the formula is a logical truth

Counterexamples

How to Formulate Them

(7) $\neg\exists x\,\mathsf{LeftOf}(x, x)$

- Let's provide a counterexample to this

1. Replace predicates & names w/non-sense ones:
   (8) $\neg\exists x\,\mathsf{Glirs}(x, x)$
2. Try to reinterpret the non-sense and find a circumstance under which the reinterpreted formula is false:
   - Let Glirs mean *loves*
   - As a matter of fact $\mathsf{Loves}(\mathsf{tom.cruise}, \mathsf{tom.cruise})$
   - In this case $\neg\exists x\,\mathsf{Loves}(x, x)$ is false
   - Therefore (7) is not a logical truth!

FO Validity

The Replacement Method for FO Validities

The Replacement Method (FO Validities)

The following method can be used to check whether or not S is a FO Validity

1. Systematically replace all of the predicates, other than =, and names with new, meaningless predicates and names
2. Try to describe a circumstance, along with interpretations for the names and predicates, in which S is false.
   - If there is no such circumstance and interpretation, S is a FO validity
   - If there is such a circumstance and interpretation, it's called a *counterexample* and S is not a FO validity

FO Validty
One More Example

(9) $\forall x\,(\mathsf{Larger}(x,a) \rightarrow \mathsf{Smaller}(a,x))$

1. Replace predicates and names with non-sense:
   (9′) $\forall x\,(\mathsf{Lirrs}(x,alf) \rightarrow \mathsf{Stams}(alf,x))$
2. Try to assign a meaning to the non-sense and construct a circumstance in which (7′) is false:
   - Let Lirrs mean *dates* and Stams mean *likes*
   - Consider the following circumstance: Alf dates Bea, but Alf doesn't like her
   - So $\neg(\mathsf{Lirrs}(bea,alf) \rightarrow \mathsf{Stams}(alf,bea))$
   - Thus, $\forall x\,(\mathsf{Lirrs}(x,alf) \rightarrow \mathsf{Stams}(alf,x))$ is false

- So (9) is not a logical truth

FO Validity
Fitch

- Fitch also provides a tool for studying FO Validities (FO Logical Truths)

FO Con

- **FO Con** is like **Ana Con**, except it looks only at the meanings of the logical symbols
- You can test if a sentence is a FO Validity by seeing if it follows from no premises using **FO Con**
- Let's look at a few examples of this in Fitch (Exercises 10.24 & 10.27)

The Replacement Method
Discussion

- The replacement method is nice and all, but it doesn't seem very precise
- We just search for interpretations and circumstances and if we can't do it, it's a logical truth?
  - No. There is an objective fact of the matter about whether or not it can be done
- Although this search seems hazy and unstructured, it can be made much more precise
  - This would involve learning a branch of mathematics called *model theory*, which is beyond our aspirations in this class
  - Chapter 18 of *LPL* uses model theory to make the replacement method more precise

The Replacement Method
Discussion

- The replacement method provides an analysis of logical possibility
- This analysis can also be applied to making the idea of logical consequence more precise
- This was another one of Alfred Tarski's innovations
- So, let's learn how to use the replacement method to test for logical consequence

Introducing
FO Consequence

Logical Consequence

C is a logical consequence of $P_1, \ldots, P_n$ iff it is **impossible** for $P_1, \ldots, P_n$ to be true while C is false

- *Impossible* means **logically** impossible
- A logical possibility can be analyzed as pair consisting of a circumstance (state of the world) and a reinterpretation of the nonlogical symbols

FO Consequence

C is a FO Consequence of $P_1, \ldots, P_n$ iff in every circumstance and under every reinterpretation of the non-logical symbols, if $P_1, \ldots, P_n$ come out true, C does too

FO Consequence
An Example

Argument 1

$\forall x\,(\mathsf{Small}(x) \to \mathsf{Cube}(x))$
$\mathsf{Small}(a)$

$\mathsf{Cube}(a)$

Argument 1′

$\forall x\,(\mathsf{Nar}(x) \to \mathsf{Wiv}(x))$
$\mathsf{Nar}(n)$

$\mathsf{Wiv}(n)$

- Let's see if we can find a circumstance and reinterpretation of Argument 1 that makes the premises true and the conclusion false
- *All nars are wivs,* ***b*** *is a nar, so* ***n*** *is a wiv*
- This still sounds valid, whatever nars, wivs and *n* are

- So, Cube(a) is a FO Consequence of the premises

FO Consequence
A Different Example

Argument 2

$\mathsf{Cube}(a)$
$\mathsf{Dodec}(b)$

$\neg(a = b)$

Argument 2′

$\mathsf{Rah}(n)$
$\mathsf{Bru}(m)$

$\neg(n = m)$

- So, $\neg(a = b)$ is not a FO Consequence of the premises

- Let's see if we can find a circumstance and reinterpretation of Argument 1 that makes the premises true and the conclusion false
- Let Rah mean *is a reporter*, Bru mean *is a super-hero*, n mean *Clark Kent* and m mean *Superman*
- Now consider the fictional world of the superman comics:
  - Rah(n) is true
  - Bru(m) is true
  - But $\neg(n = m)$ is false

FO Consequence
The Replacement Method

The Replacement Method (FO Consequence)

The following method can be used to check whether or not C is a FO Consequence of $P_1, \ldots, P_n$:

1. Systematically replace all of the non-logical symbols with non-sense symbols
2. Try to describe a circumstance, along with interpretations of the predicates in which $P_1, \ldots, P_n$ are true and C false.
   - If there is no such circumstance and interpretation, C is a FO Consequence of $P_1, \ldots, P_n$
   - If there one, it's called a *counterexample* and C is not a FO Consequence of $P_1, \ldots, P_n$

In Class Exercise

Break into two groups. One group should do 10.10, the other 10.13.

Let's use **FO Con** in Fitch to check our answers

FO Equivalence

One Last Thing

First-Order Equivalence (FO Equivalence)

A and B are FO equivalent iff B is a FO consequence of A and A is a FO consequence of B

- So, there's nothing more to FO equivalence than to FO consequence
- To show FO consequence you just use the replacement method to show that A and B are FO consequences of each other