Translating with Quantifiers

From English to ∀ and ∃

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Announcements

03.26

- Something

Outline

Satisfaction

The Basic Idea

- Remember truth tables don't allow us to analyze the meaning of quantified sentences
- Instead, we use Alfred Tarski's (1936) idea of an object satisfying a formula
- Here's the intuition behind satisfaction
  - Although a formula with a free variable like Cube(x) is neither true nor false, we can think of it being true of some object o
  - Tarski called this special idea of being true of an object *satisfaction*
  - For example, o satisfies Small(x) ∧ Cube(x) iff o is a small cube

Satisfaction

The Precise Definition

Definition of Satisfaction

An object *o* satisfies a wff $\mathsf{S}(\mathsf{x})$ containing x as its only **free variable** iff the following two conditions are met:

1. If we give a *o* a name that's not in use, call it $\mathsf{n_i}$, then $\mathsf{S}(\mathsf{n_i})$ is true
2. $\mathsf{S}(\mathsf{n_i})$ is the result of replacing every occurrence of x in $\mathsf{S}(\mathsf{x})$ with $\mathsf{n_i}$

- Let's work through a quick example in Tarski's World

Existential Statements

When are They True?

- Given the idea of satisfaction, we can say when quantified statements are true
- Before we review the semantics for ∃, let's review the intuitive meaning of existential statements
- *Something is strange* is true iff there is some object *o* and *o* is strange
- The truth of $\exists x\, \mathsf{Strange}(x)$ can be determined in a similar way:
  - $\exists x\, \mathsf{Strange}(x)$ is true iff **some** object *o* satisfies $\mathsf{Strange}(x)$
  - That is, if there is some object *o* such that when you give it an unused name n, $\mathsf{Strange}(\mathsf{n})$ comes out true
  - If there is no such object, $\exists x\, \mathsf{Strange}(x)$ is false

Existential Statements

Official Semantics

Semantics for ∃

$\exists x\, \mathsf{S}(x)$ is true iff there is at least one object that satisfies $\mathsf{S}(x)$

Example

When is $\exists x\, (\mathsf{Large}(x) \land \mathsf{Tet}(x))$ true?

- By the semantics for ∃:
  - (1) Iff there is at least one object that satisfies $\mathsf{Large}(x) \land \mathsf{Tet}(x)$
- By the definition of satisfaction (1) amounts to:
  - Iff when we give *o* some unused name n, $\mathsf{Large}(\mathsf{n}) \land \mathsf{Tet}(\mathsf{n})$ comes out true

Existential Statements

The Game Rule for ∃

Game Rule for ∃

Given $\exists x\, \mathsf{S}(x)$:

Your Commitment	Player to Move	Goal
TRUE	you	Choose some *o* that satisfies $\mathsf{S}(x)$
FALSE	Tarski's World	

- $\mathsf{S}(x)$ is any wff containing a free occurrence of x:
  - $\mathsf{Cube}(x)$
  - $\mathsf{Cube}(x) \land \exists y\, \mathsf{Small}(y)$
  - $\neg(\forall y\, \mathsf{Tet}(y) \rightarrow (\mathsf{Small}(x) \lor \mathsf{Cube}(a)))$
- Let's play some games in Tarski's World!

Universal Statements
When are They True?

- When are universal statements are true?
- Before we review our precise answer, let's recall some basic intuitions
- *Everything is on fire* is true iff for every object o, o is on fire
- The truth of $\forall x\, \mathsf{OnFire}(x)$ can be determined in a similar way:
  - Consider whether every object o in the domain of discourse satisfies $\mathsf{OnFire}(x)$
  - That is, for every object o see whether when you give it an unused name n, $\mathsf{OnFire}(\mathsf{n})$ comes out true
  - If so, then $\forall x\, \mathsf{OnFire}(x)$ is true
  - Otherwise, it is false
- Okay, let's see that precise definition

Universal Statements
Official Semantics

Semantics for ∀

$\forall x\, \mathsf{S}(x)$ is true iff every object satisfies $\mathsf{S}(x)$

Example

When is $\forall x\, (\mathsf{Cube}(x) \land \mathsf{Small}(x))$ true?

- By the semantics for ∀:

  (2) Iff every object o satisfies $\mathsf{Cube}(x) \land \mathsf{Small}(x)$
- By the definition of satisfaction (2) amounts to:
  - Iff when we give each o some unused name n, $\mathsf{Cube}(\mathsf{n}) \land \mathsf{Small}(\mathsf{n})$ comes out true

- Let's go to Tarski's World and evaluate some universal claims

Universal Statements
The Game Rule for ∀

Game Rule for ∀

Given $\forall x\, \mathsf{S}(x)$:

Your Commitment	Player to Move	Goal
TRUE	Tarski's World	Choose some o that does not satisfy $\mathsf{S}(x)$
FALSE	you	

- As always $\mathsf{S}(x)$ is any wff containing a free occurrence of x
- Let's play some games in Tarski's World!

Semantics for the Quantifiers
Summary

- We have learn two methods for understanding the meaning of ∀ and ∃:
  1. Our satisfaction-based definition of when $\forall\, \mathsf{S}(x)$ and $\exists x\, \mathsf{S}(x)$ are true
  2. Our game-rule definition, which says how committing to the truth or falsity of a quantified formula affects a game based on that formula
- We just saw the deep parallel in these two methods
- The game just carries you through the steps you'd go through if you applied the semantics for ∀ or ∃ and then the definition of satisfaction

The Four Aristotelian Forms
What they Are

The Four Aristotelian Forms

1. *All A's are B's*
2. *Some A's are B's*
3. *No A's are B's*
4. *Some A's are not B's*

- These are four of the most common quantificational sentences used in quantificational reasoning
- We can represent all of them in FOL now that we have ∀ and ∃
- Today, we'll learn how!

The First Aristotelian Form
All A's are B's

The Form: *All A's are B's*

(3) *All rabbits are vicious*

Paraphrase For every x, if x is a rabbit then x is vicious

Translation ∀x (Rabbit(x) → Vicious(x))

- This translation has the form: ∀x (A(x) → B(x))

General Fact

All A's are B's translates as ∀x (A(x) → B(x))

The Second Aristotelian Form
Some A's are B's

The Form: *Some A's are B's*

(4) *Some professors are vicious*

Paraphrase Some thing x is both professor and vicious

Translation ∃x (Professor(x) ∧ Vicious(x))

- This translation has the **form**: ∃x (A(x) ∧ B(x))

General Fact

Some A's are B's translates as ∃x (A(x) ∧ B(x))

The Second Aristotelian Form
Comments

- We've learned two facts:
  1. *All As are Bs* translates as ∀x (A(x) → B(x))
  2. *Some As are Bs* translates as ∃x (A(x) ∧ B(x))
- Why don't we translate *Some As are Bs* as ∃x (A(x) → B(x))?
- We'll see this by doing exercise **9.8**

The Third Aristotelian Form
No A's are B's

The Form: *No A's are B's*

(5) *No students are drunk*

Paraphrase 1 For every x, if x is a student then x is not drunk

Paraphrase 2 It is not the case that for some x, x is a student and x is drunk

Translation 1 $\forall x\,(\mathsf{Student}(x) \rightarrow \neg \mathsf{Drunk}(x))$

Translation 2 $\neg \exists x\,(\mathsf{Student}(x) \wedge \mathsf{Drunk}(x))$

- Translation 1 has the form: $\forall x\,(A(x) \rightarrow \neg B(x))$
- Translation 2 has the form: $\neg \exists x\,(A(x) \wedge B(x))$
- These are equivalent, and we'll eventually prove it

The Third Aristotelian Form
No A's are B's (Continued)

General Fact

No A's are B's translates as:

$$\forall x\,(A(x) \rightarrow \neg B(x))$$

Or:

$$\neg \exists x\,(A(x) \wedge B(x))$$

The Fourth Aristotelian Form
Some A's are not B's

The Form: *Some A's are not B's*

(6) *Some excuses are not believable*

Paraphrase For some x, x is an excuse and x is not believable

Translation $\exists x\,(\mathsf{Excuse}(x) \wedge \neg \mathsf{Believable}(x))$

- This translation has the form: $\exists x\,(A(x) \wedge \neg B(x))$

General Fact

Some A's are not B's translates as $\exists x\,(A(x) \wedge \neg B(x))$

The 4 Aristotelian Forms
Summary

The Aristotelian Forms and Their Translations

All A's are B's	$\forall x\,(A(x) \rightarrow B(x))$
Some A's are B's	$\exists x\,(A(x) \wedge B(x))$
No A's are B's	$\forall x\,(A(x) \rightarrow \neg B(x))$
Some A's are not B's	$\exists x\,(A(x) \wedge \neg B(x))$

Beyond the Second Form
What to Do

- Translate:

  (7) *Some cubes are in front of* ***c***
- It has the second form: *Some A's are B's*. So:

$$\exists x\,(\mathsf{Cube}(x) \land \mathsf{FrontOf}(x, b))$$

- What about:

  (8) *Some small cubes are in front of* ***c***

  That's not one of the forms we know!
- Still, it's pretty obvious how it should go:

$$\exists x\,(\mathsf{Small}(x) \land \mathsf{Cube}(x) \land \mathsf{FrontOf}(x, b))$$

Beyond the Second Form
Multiply Restricted Existentials

- From the second form, we know that you restrict ∃ with ∧
- An existential quantifier multiply restricted means multiple conjuncts restricting ∃:

(9) *Some cute little kitten ate Alex*

$$\exists x\,(\mathsf{Cute}(x) \land \mathsf{Little}(x) \land \mathsf{Kitten}(x) \land \mathsf{Ate}(x, alex))$$

(10) *A small rat scared Jay*

$$\exists x\,(\mathsf{Small}(x) \land \mathsf{Rat}(x) \land \mathsf{Scared}(x, jay))$$

(11) *At least one small cube in front of* ***b*** *is left of* ***c***

$$\exists x\,(\mathsf{Small}(x) \land \mathsf{Cube}(x) \land \mathsf{FrontOf}(x, b) \land \mathsf{LeftOf}(x, c))$$

Beyond the First Form
What to Do?

- Translate:

  (12) *All cubes are in front of* ***c***
- It's form is *All A's are B's*, so:

$$\forall x\,(\mathsf{Cube}(x) \rightarrow \mathsf{FrontOf}(x, b))$$

- What about:

  (13) *All small cubes are in front of* ***c***
- That's not one of the forms we know!

Beyond the First Form
What to Do

- We know that you restrict ∀ with → (1st Form)
- A universal quantifier multiply restricted means multiple restrictions of ∀ with →:

(14) *All cute little kittens hate Alex*

$$\forall x\,(\mathsf{Cute}(x) \rightarrow (\mathsf{Little}(x) \rightarrow (\mathsf{Kitten}(x) \rightarrow \mathsf{Hate}(x, alex))))$$

(15) *Every small rat scared Jay*

$$\forall x\,(\mathsf{Small}(x) \rightarrow (\mathsf{Rat}(x) \rightarrow \mathsf{Scared}(x, jay)))$$

(16) *Every small cube in front of* ***b*** *is left of* ***c***

$$\forall x\,(\mathsf{Small}(x) \rightarrow (\mathsf{Cube}(x) \rightarrow (\mathsf{FrontOf}(x, b) \rightarrow \mathsf{LeftOf}(x, c))))$$

Beyond the First Form
Using ∧ Instead of →

- Instead of nesting →, you can use conjoin the restrictions into one:

$$\forall x\,(\mathsf{Cute}(x) \to (\mathsf{Little}(x) \to (\mathsf{Kitten}(x) \to \mathsf{Hate}(x, alex))))$$

Is Equivalent to:

$$\forall x\,((\mathsf{Cute}(x) \land \mathsf{Little}(x) \land \mathsf{Kitten}(x)) \to \mathsf{Hate}(x, alex))$$

- This is because of the following general equivalence:

$$A \to (B \to C) \iff (A \land B) \to C$$

Subjects and Objects
Some Terminology

- Some predicates like *love* relate two things:
  (17) *Kay loves Jay*
- When you have a predicate that relates two things, it's helpful to have some terminology to distinguish those two things
- *Kay* is the subject
- *Jay* is the object
- Intuitively, the subject is what the sentence is primarily about

Roaming Quantifiers
In Object Position

- So far, we've only considered sentences with quantifiers in subject-position:
  (18) *Every cube is in front of* ***b***
- What about when you have a quantifier in object-position?
  (19) ***b*** *is in front of everything*
- Just stick ∀ out in front of the predicate, and 'quantify into' the object position

$$\forall x\, \mathsf{FrontOf}(b, x)$$

Roaming Quantifiers
More on Object Position

- Okay, but what happens when the quantifier in object position is restricted
  (20) ***b*** *is in front of every cube*
- You have to move its restrictor out front too:
  (20′) $\forall x\,(\mathsf{Cube}(x) \to \mathsf{FrontOf}(b, x))$
- This holds for multiply restricted ones too:
  (21) ***b*** *is in front of every small cube*
  Translates as:
  (21′) $\forall x\,((\mathsf{Cube}(x) \land \mathsf{Small}(x)) \to \mathsf{FrontOf}(b, x))$

Roaming Quantifiers

Some More Examples

(22) shows that you move the restrictors to the left of the predicate, but no further!

(22) a. *It's not the case that **b** is a large cube*
b. $\neg\exists y\,(\mathsf{Large}(y) \wedge \mathsf{Cube}(y) \wedge b = y)$

(23) a. *It's not the case that something is a large cube*
b. $\neg\exists y\,(\mathsf{Large}(y) \wedge \mathsf{Cube}(y) \wedge \exists x\, x = y)$

(24) a. *Everything between **c** and **b** is **a***
b. $\forall x\,(\mathsf{Between}(x, c, b) \rightarrow x = a)$

(25) a. *Everything between **c** and **b** is a cube*
b. $\forall x\,(\mathsf{Between}(x, c, b) \rightarrow \exists y\,(\mathsf{Cube}(y) \wedge x = y))$

An Oddity

Existentials in Conditionals

- Consider:
  (26) *If a yokel drools, he snores*
- *a* is existential, right?
- So, it seems like we should translate (26) as:
  (27) $\exists x\,((\mathsf{Yokel}(x) \wedge \mathsf{Drools}(x)) \rightarrow \mathsf{Snores}(x))$
- This requires at least one yokel that drools to snore
- Is that strong enough?

An Oddity

Existentials in Conditionals are Universal?

- Most people get the intuition that:
  (26) *If a yokel drools, he snores*
  Is equivalent to:
  (28) *Every yokel who drools snores*
- But then (26) shouldn't be translated with ∃ as in (27), but rather:
  (29) $\forall x\,((\mathsf{Yokel}(x) \wedge \mathsf{Drools}(x)) \rightarrow \mathsf{Snores}(x))$
- So, beware, in conditionals, existentials sound like universals