Basics of Quantification

$\forall$ and $\exists$

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Announcements

03.24

1. Something

Outline

Quantities

In Thought & Talk

- In our daily lives, we think & talk about quantities
  - Some money
  - Every ex-girlfriend
  - Two siblings
  - No friends
  - Many friends
- As it turns out, this thought & talk is governed by interesting logical principles
- These logical principles cannot be captured with the truth-functional connectives

Quantifiers

And Quantifier Phrases

(1) Some money is wasted

(2) Every magician is a vampire

(3) Two cats are meowing

(4) No friends showed up to George's party

(5) Many friends came to my party

- The above sentences contain quantifier phrases
- Simple quantifier phrases have two parts:
  1. A quantifier
  2. A noun
- Last class, we learned how to represent quantifiers and quantifier phrases in FOL

Quantifiers in FOL

Meet ∀ and ∃

- We added the quantifier symbols for FOL:
  The Universal Quantifier ∀ (*everything*)
  The Existential Quantifier ∃ (*something*)
- And variables
  - FOL has infinitely many variables:
    $t, u, v, w, x, y, z, t_1, \ldots, t_n, u_1, \ldots, u_n, v_1, \ldots, v_n, \ldots$
  - They go in the slots of predicates:
    $\mathsf{Cube}(y), \mathsf{FrontOf}(u, v), \mathsf{Between}(z, u_{21}, w)$
- Together, these two resources allowed us to represent quantificational sentences

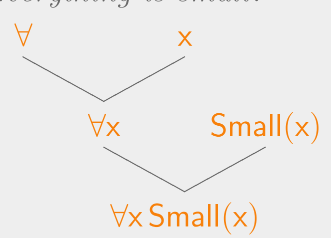
The Universal Quantifier

Universal Statements

- How do you represent a universal statement in FOL?

Example

Everything is small:



1. It's a universal statement, so use ∀
2. Pick a variable to use, like x
3. Pair ∀ with that variable
4. Plug that variable into the predicate of the claim
5. Stick together the two things you've made

- We read $\forall x\, \mathsf{Small}(x)$ as *For every object x, x is small*
- This is an intuitively correct paraphrase of *Everything is small*

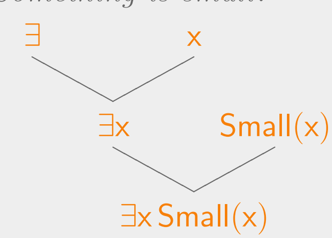
The Quantifier

Existential Statements

- How do you represent a Existential statement in FOL?

Example

Something is small:



1. Its a existential statement, so use ∃
2. Pick a variable to use, like x
3. Pair ∃ with that variable
4. Plug that variable into the predicate of the claim
5. Stick together the two things you've made

- We read $\exists x\, \mathsf{Small}(x)$ as *For some object x, x is small*
- This is an intuitively correct paraphrase of *Something is small*

Variables

Complete vs. Incomplete

- There's a big difference between these two formulas:

  (6) Small(x)

  (7) Small(a)

- (7) makes a claim that is true or false
  - Either *a* is small or it isn't
- (6) does not
- (6) is an incomplete claim
  - It's like saying *it is small* without telling us what *it* is!
- However, (6) becomes complete when ∃x or ∀x is added
- '∃x Small(x)' **is** either true or false

Variables

Complete vs. Incomplete?

A Question

When exactly does a formula containing variables make a complete claim?

- Does (8) make a complete claim?

  (8) ∃x (Small(x) ∧ Cube(x))

- What about (9)?

  (9) ∃x (Small(x) ∧ Cube(x)) ∨ LeftOf(x, a)

The Answer (First Version)

A formula containing variables makes a complete claim just in case every variable appears within the scope of a quantifier symbol attached to that variable

Scope

Some Terminology

Scope

1. A quantificational wff ∀v A is formed by sticking together some wff A and quantifer-phrase ∀v
2. We call A that quantifier's *scope*.

- ∀x (Small(x) ∧ Tet(x))
  - ∀x's Scope: Small(x) ∧ Tet(x)
- ∀x Small(x) ∧ Tet(x)
  - ∀x's Scope: Small(x)

Variables

Complete vs. Incomplete: Revisited

The Answer (First Version)

A formula containing variables makes a complete claim just in case every variable appears within the scope of a quantifier symbol attached to that variable

- Does (8) make a complete claim?

  (8) ∃x (Small(x) ∧ Cube(x))

  - Yes; both occurrences within scope of ∃x
- What about (9)?

  (9) ∃x (Small(x) ∧ Cube(x)) ∨ LeftOf(x, a)

  - No; 3rd occurrence outside scope of ∃x

Binding

More Terminology

Bondage

An occurrence of a variable v is bound iff v occurs within the scope of either ∀v or ∃v

- 1st & 2nd occurrences of x are bound; 3rd is not
  (9) ∃x (Small(x) ∧ Cube(x)) ∨ LeftOf(x, a)

Freedom

An occurrence of a variable v is free iff v does **not** occur within the scope of either ∀v or ∃v

- 3rd occurrence of x in (9) is free; 1st & 2nd are not

Two More Things

A New Version of The Answer & Wffs vs. Sentences

The Answer (Second Version)

A formula containing variables makes a complete claim just in case every variable is bound

Sentences vs. Wffs (Approximation)

1. *Well-formed formulas* or *wffs* is the set of **all** grammatical expressions of FOL, including both incomplete claims, like 'Tet(x)' and complete ones
2. *Sentences* are formulas that make complete claims; contain no variables or only bound ones

Wffs v. Non-Wffs

Some Examples

Wffs

(10) Tet(a)
(11) Cube(y)
(12) (Cube(y) ∧ Tet(a))
(13) (∃y (Cube(y) ∧ Tet(a)))
(14) (∃y Cube(y)) ∧ Tet(a)
(15) Tet(a) → (Cube(b) ∧ Small(b))

Non-Wffs

(16) Tet
(17) (y)Cube
(18) Cube(y, Small)
(19) ∧Cube(y) Tet(a)
(20) ∃(Cube(y) ∧ Large(y))
(21) Tet(a) → Cube(b) ∧ Small(b)

- Now that we're clear on the wff v. non-wff distinction, let's draw the one we set out to draw
  - The wff v. sentence distinction

Sentences v. Wffs

Some Examples

Non-Sentence Wffs

(22) Tet(y)
(23) ¬Cube(y)
(24) (Cube(y) ∧ Tet(a))
(25) ((∃y Cube(y)) ∧ Tet(y))
(26) (∃y (Cube(y) ∧ Tet(x)))

- Free variables

Sentences

(27) Tet(a)
(28) ¬Tet(a)
(29) (Cube(a) ∧ Tet(a))
(30) (∃y (Cube(y) ∧ Tet(y)))
(31) (∃y (Cube(y) ∧ (∃x Tet(x))))

- No free variables

Semantics & Quantification

Where we Are

- We know what the truth functional connectives ($\wedge, \vee, \neg, \rightarrow, \leftrightarrow$) mean
  - Their meanings are given by their truth tables
  - **Terminology**: *semantics* is the study of **meaning**
- We have not yet learned the semantics for quantifier symbols ($\forall$, $\exists$)
- As it turns out, we cannot provide a semantics for quantifiers using **truth tables**
- Why?

Semantics & Quantification

Why Not Truth Tables

- Truth tables work by explaining the truth of a complex formula in terms of the truth of its parts
  - Example: ¬P is T iff P is F
- The problem with using truth tables for quantifiers is that the truth of quantified formulas cannot be determined from the truth of its parts
  - Example: ∀x Cube(x) is T iff ???
    - Cube(x) is T? F?
    - Neither!
    - Cube(x) isn't capable of truth or falsity, it's too incomplete!
- So, we can't use truth tables to explain what quantified sentences mean

Satisfaction

The Basic Idea

- If not truth tables, what?
- We'll use a method pioneered by Alfred Tarski (1936)
- He introduced the idea of an object satisfying a formula
- Here's the intuition behind satisfaction
  - Although a formula with a free variable like Cube(x) is neither true nor false, we can think of it being true of some object *o*
  - Tarski called this special idea of being true of an object *satisfaction*
  - For example, *o* satisfies Small(x) ∧ Cube(x) iff *o* is a small cube

Satisfaction

The Precise Definition

Definition of Satisfaction

An object *o* satisfies a wff S(x) containing x as its only **free variable** iff the following two conditions are met:

1. If we give a *o* a name that's not taken, call it n_i, then $S(n_i)$ is true
2. $S(n_i)$ is the result of replacing every occurrence of x in S(x) with n_i

- Let's work through some examples in Tarski's World

Domain of Discourse
The Things We're Talking About

- When we ask:
  - Is there an object o that satifies $\mathsf{S}(x)$?
- Which objects should we look at?
- When we communicate, we take as given a collection of objects we're interested in talking about
- Sometimes this collection is absolutely all objects, but more commonly it is some restricted set of objects
- We'll call this set the *domain of discourse*
- So, the answer to our question above is: *the objects in the domain of discourse!*

Domain of Discourse
An Example

Example

- When I say *Every student is sleepy* here and now, which students does it seem most reasonable for me to be talking about?
- You! The students in this classroom (Sadly)
- The domain of discourse is taken to be set of things in this room
- When I say *every student* I restrict your attention to the students in this room

- In Tarski's World the domain of discourse is the collection of blocks on the board

Existential Statements
When are They True?

- Now that we understand satisfaction, we can say when quantified statements are true
- Before we look at the exact definitions, let's get some intuitions clear
- *Something is smelly* is true iff there is some object o and o is smelly
- The truth of $\exists x\, \mathsf{Smelly}(x)$ can be determined in a similar way:
  - $\exists x\, \mathsf{Smelly}(x)$ is true iff **some** object o satisfies $\mathsf{Smelly}(x)$
  - That is, if there is some object o such that when you give it an unused name n, $\mathsf{Smelly}(\mathsf{n})$ comes out true
  - If there is no such object, $\exists x\, \mathsf{Smelly}(x)$ is false

Existential Statements
Official Semantics

Semantics for ∃

$\exists x\, \mathsf{S}(x)$ is TRUE iff there is at least one object that satisfies $\mathsf{S}(x)$

Example

When is $\exists x\, (\mathsf{Cube}(x) \land \mathsf{Small}(x))$ true?

- By the semantics for ∃:

  (32) Iff there is at least one object that satisfies $\mathsf{Cube}(x) \land \mathsf{Small}(x)$
- By the definition of satisfaction (33) amounts to:
  - Iff when we give o some unused name n, $\mathsf{Cube}(\mathsf{n}) \land \mathsf{Small}(\mathsf{n})$ comes out true

Existential Statements
Examples

- The way to understand these definitions is by going through examples
- Let's go to Tarski's World and evaluate some existential claims

Universal Statements
When are They True?

- When are universal statements are true?
- Before we look at the exact definition, let's get some intuitions clear
- *Everything is beautiful* is true iff for every object o, o is smelly
- The truth of $\forall x\,\mathsf{Beautiful}(x)$ can be determined in a similar way:
  - Consider whether every object o in the domain of discourse satisfies $\mathsf{Beautiful}(x)$
  - That is, for every object o see whether when you give it an unused name n, $\mathsf{Beautiful}(\mathsf{n})$ comes out true
  - If so, then $\forall x\,\mathsf{Beautiful}(x)$ is true
  - Otherwise, it is false
- Okay, let's see the precise definition

Universal Statements
Official Semantics

Semantics for $\forall$

$\forall x\,\mathsf{S}(x)$ is TRUE iff every object satisfies $\mathsf{S}(x)$

Example

When is $\forall x\,(\mathsf{Cube}(x) \wedge \mathsf{Small}(x))$ true?

- By the semantics for $\forall$:

  (33) Iff every object o satisfies $\mathsf{Cube}(x) \wedge \mathsf{Small}(x)$
- By the definition of satisfaction (33) amounts to:
  - Iff when we give each o some unused name n, $\mathsf{Cube}(\mathsf{n}) \wedge \mathsf{Small}(\mathsf{n})$ comes out true

Universal Statements
Examples

- The way to understand these definitions is by going through examples
- Let's go to Tarski's World and evaluate some universal claims