

# Conditionals

## The Basics

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# Outline

- 1 Introduction
- 2 Meaning
- 3 Translation
- 4 Conclusion

# Announcements

For 02.26

- 1 There's **free logic tutoring** available from the Rutgers Learning Centers
  - Logic tutoring is done in the Kreeger Learning Center (College Ave.) from 6-8 on Sunday & 5-7 on Monday
  - For more details see the recent Sakai announcement
- 2 The midterm is a week from next Tuesday
  - HW7 will be a practice midterm
  - It will have the exact format of the real midterm

# Introduction

Beyond the Booleans

- We've learned how to treat the Boolean connectives (*and*, *or*, *not*) in FOL
  - Using the connectives  $\wedge, \vee, \neg$  we've learned:
    - 1 How to translate sentence of English with *and*, *or* and *not* into FOL
    - 2 What these sentences mean, using **truth tables** and **proofs** for  $\wedge, \vee, \neg$
- But the Booleans are just the tip of the iceberg!
- There are many other logically important constructions in natural language that we need to learn to treat in FOL

## Introduction

## Other Connectives

- Consider these sentences:
  - (1) Herb is wearing fake alligator boots **if** he is going to the market
  - (2) Josh is surfing **only if** he is in California
  - (3) Maria is doing math **if and only if** she is in NYC
  - (4) Jerry will go to the opera **unless** it is Verde
  - (5) Susan quit **because** she found a better career
- These all contain a **logical connective** you haven't studied yet

## Introduction

## Today's Topics

- Today we are going to learn how to treat 4 of those 5 connectives (and a few others) in FOL
- The 4 we will cover:
  - ① *if*
  - ② *only if*
  - ③ *if and only if*
  - ④ *unless*
- These can be treated in FOL with only two new connectives:  $\rightarrow$ ,  $\leftrightarrow$
- Why are we ignoring *because*?

## Introduction

## Truth-Functionality

- *Because* is not **truth functional**
- What does *truth functional* mean?
  - Suppose we have some binary sentential connective  $\star$
  - To say that  $\star$  is truth functional is just to say that the truth value of  $A \star B$  is completely determined by the truth values of  $A$  and  $B$
  - $\wedge$  is truth functional:
    - $A \wedge B$  is true if and only if  $A$  is true and  $B$  is true, and false otherwise
- To see exactly why *because* is not truth functional read pp.177 of *LPL*
- Non-truth functional connectives are **not defective**
  - They are just very hard to analyze and require more sophisticated tools than those taught in this class

## Introduction

## The Plan for Today

- Today we'll add two new connectives to FOL:  $\rightarrow$  and  $\leftrightarrow$
- We'll learn what they **mean** in terms of **truth tables** and **game rules**
- Then we'll learn how to use them to **translate** sentences containing *if*, *only if*, *if and only if*, *unless* and a few other English connectives

# The Basics

## Grammar and Terminology

- The  $\rightarrow$  symbol is used to combine two sentences P and Q to form a new sentence:  
(6)  $P \rightarrow Q$
- A sentence like (6) is called a **material conditional**
- P is called the **antecedent**
- Q is called the **consequent**
- We will discuss the English counterparts of  $\rightarrow$  after we learn what it means
- For now, we'll read  $P \rightarrow Q$  as *If P then Q*

# The Basics

## The Meaning of $\rightarrow$

### Truth Table for $\rightarrow$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- $P \rightarrow Q$  is T when the truth of P somehow guarantees that Q will be T
- $P \rightarrow Q$  is F when P is T and Q is F
- Otherwise,  $P \rightarrow Q$  is T

### Game Rule for $\rightarrow$

When playing a game Tarski's World treats  $P \rightarrow Q$  as an abbreviation for  $\neg P \vee Q$ . This means:

- 1 If you commit to the truth of  $P \rightarrow Q$ , you commit to the falsity of P or the truth of Q
- 2 If you commit to the falsity of  $P \rightarrow Q$ , you commit to the truth of P and the falsity of Q

# The Basics

## More on the Meaning of $\rightarrow$

### Truth Table for $\rightarrow$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- This table for  $\rightarrow$  makes  $P \rightarrow Q$  tautologically equivalent to  $\neg P \vee Q$
- Let's see this in detail using Boole to construct a joint truth table
- This table also says that  $P \rightarrow Q$  is F just in case P is T and Q is F
  - So  $\neg(P \rightarrow Q)$  and  $P \wedge \neg Q$  should be equivalent
  - Again, let's see what Boole says!
- Now let's look at some sentences in Tarski's World to see if we understand when material conditionals are true

# The Basics

## Grammar and Terminology

- The  $\leftrightarrow$  symbol is used to combine two sentences P and Q to form a new sentence:  
(7)  $P \leftrightarrow Q$
- A sentence like (7) is called a **material biconditional**
- The most common way to read (7) is:  
*P if and only if Q*
- It is common to abbreviate *if and only if* as *iff*
- Now, what does  $\leftrightarrow$  mean?

# The Basics

The Meaning of  $\leftrightarrow$

## Truth Table for $\leftrightarrow$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

- $P \leftrightarrow Q$  is T when P and Q have the same truth value
- $P \leftrightarrow Q$  is F when P and Q have different truth values

## Game Rule for $\leftrightarrow$

When playing the game, Tarski's World treats  $P \leftrightarrow Q$  as an abbreviation for  $(P \rightarrow Q) \wedge (Q \rightarrow P)$ . This means that committing to the truth of  $P \leftrightarrow Q$  commits you to the truth of... something complex. But it is equivalent to committing yourself to P and Q having the same truth value.

# The Basics

More on the Meaning of  $\leftrightarrow$

## Truth Table for $\leftrightarrow$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

- This table for  $\leftrightarrow$  makes  $P \leftrightarrow Q$  tautologically equivalent to  $(P \rightarrow Q) \wedge (Q \rightarrow P)$
- We'll show it w/Boole
- This table also says that  $P \leftrightarrow Q$  is T just in case P and Q have the same truth values
  - So  $P \leftrightarrow Q$  and  $(P \wedge Q) \vee (\neg P \wedge \neg Q)$  should be equivalent; let's turn to Boole again!
- Let's look at some sentences in Tarski's World to see if we understand when material biconditionals are true

# Conditionals

In-Class Exercise

*A world and some sentences will be displayed in Tarski's World. It is your job to correctly determine the truth value of each sentence in the world displayed.*

# Translation with $\rightarrow$

Reading  $P \rightarrow Q$

- Perhaps the most natural way to read:
  - (8)  $P \rightarrow Q$
  - is:
  - (8') *If P then Q*
- So
  - (9)  $\text{Tet}(a) \rightarrow \text{Smaller}(a, b)$
  - Could be read as:
  - (9') *If a is a tetrahedron, then a is smaller than b*
- Remember that  $\rightarrow$  can be used to represent English connectives other than *if... then*

## Translation with $\rightarrow$

### Varieties of Conditionals

- We've already seen that:
  - (8') *If P then Q*
  - Gets represented in FOL as  $P \rightarrow Q$
- But so do all of the following:
  - (10) *Q if P*
  - (11) *P only if Q*
  - (12) *Q provided P*
- Also, *P unless Q* translates as  $\neg Q \rightarrow P$
- Let's look at some concrete examples of these sentences and learn how to find their correct translations

## Translation with $\rightarrow$

### *P if Q*

How should we translate:

(13) Homer drinks **if** Bart fails in school

First, recognize the form:

- (13) has the form *P if Q*
- *P if Q* translates as  $Q \rightarrow P$

Second, we translate the parts, *P* and *Q*:

- *P*: *Homer drinks*, so **P : Drinks(homer)**
- *Q*: *Bart fails in school*, so **Q : Fails(bart)**

Last, we plug our translations of *P* and *Q* into the form:

$$\begin{array}{l}
 Q \rightarrow P \\
 \checkmark \text{ Fails(bart)} \rightarrow \text{Drinks(homer)}
 \end{array}$$

## Translation with $\rightarrow$

### *P only if Q*

Translate:

(14) Homer drinks **only if** Lenny pays

First, the form:

- (14) has the form *P only if Q*
- *P only if Q* translates as  $P \rightarrow Q$

Second, we translate **the parts**:

- *P*: *Homer drinks*, so **P : Drinks(homer)**
- *Q*: *Lenny pays*, so **Q : Pays(lenny)**

Last, we plug the translation of the parts into the form:

$$\begin{array}{l}
 P \rightarrow Q \\
 \checkmark \text{ Drinks(homer)} \rightarrow \text{Pays(lenny)}
 \end{array}$$

## Translation with $\rightarrow$

### *P provided Q*

Translate:

(15) Homer drinks **provided** Moe is alive

First, the form:

- (15) has the form *P provided Q*
- *P provided Q* translates as  $Q \rightarrow P$

Second, translate **the parts**:

- *P*: *Homer drinks*, so **P : Drinks(homer)**
- *Q*: *Moe is alive*, so **Q : Alive(moe)**

Last, plug the parts into **the form**:

$$\begin{array}{l}
 Q \rightarrow P \\
 \checkmark \text{ Alive(moe)} \rightarrow \text{Drinks(homer)}
 \end{array}$$

Translation with  $\rightarrow$  $P$  unless  $Q$ 

Translate:

(16) Garfield is asleep unless he is eating lasagna

First, the form:

- (16) has the form  $P$  unless  $Q$
- $P$  unless  $Q$  translates as  $\neg Q \rightarrow P$

Second, translate the parts:

- $P$ : Garfield is sleeping, so  $P$ : Asleep(garfield)
- $Q$ : He is eating lasagna, so  $Q$ : EatingLas(garfield)

Last, plug the parts into the form:

$$\begin{array}{l} \neg Q \rightarrow P \\ \checkmark \quad \neg\text{EatingLas}(\text{garfield}) \rightarrow \text{Asleep}(\text{garfield}) \end{array}$$

Translation with  $\rightarrow$ 

Summary

English Form	FOL Translation
If $P$ then $Q$	$P \rightarrow Q$
$P$ only if $Q$	$P \rightarrow Q$
$P$ if $Q$	$Q \rightarrow P$
$P$ provided $Q$	$Q \rightarrow P$
$P$ unless $Q$	$\neg Q \rightarrow P$

## Translation Recipe (For Any Sentence, Not Just Conditionals)

- 1 Recognize the form

Example  $a$  is a cube unless it is smallThe Form:  $P$  unless  $Q \rightsquigarrow \neg Q \rightarrow P$ 

- 2 Translate the parts
- 3 Plug the parts into the form

## Translation

Conditionals Compounded

Translate:

(17)  $c$  is to the right of  $d$  only if  $d$  is either a cube or small

First, recognize the form and how it's translated:

- $P$  only if  $Q \rightsquigarrow P \rightarrow Q$

Second, translate the parts:

- $P$ :  $c$  is to the right of  $d$ , so  $P$ : RightOf( $c, d$ )
- $Q$ :  $d$  is either a cube or small  
So,  $Q$ : Cube( $d$ )  $\vee$  Small( $d$ )

Plug these translations into the form:

$$\begin{array}{l} P \rightarrow Q \\ \checkmark \quad \text{RightOf}(c, d) \rightarrow (\text{Cube}(d) \vee \text{Small}(d)) \end{array}$$

## Translation

Conditionals Compounded

(18) If  $e$  is a cube, then it's to the left of  $a$  unless  $a$  is small

Recognize the form:

- If  $P$  then  $Q \rightsquigarrow P \rightarrow Q$

Now, the parts:

- $P$ :  $e$  is a cube, so  $P$ : Cube( $e$ )
- $Q$ :  $e$  is to the left of  $a$  unless  $a$  is small

The form:

- $R$  unless  $S \rightsquigarrow \neg S \rightarrow R$

Translate the parts:

- $R$ :  $e$  is to the left of  $a$ , so  $R$ : LeftOf( $e, a$ )
- $S$ :  $a$  is small, so  $S$ : Small( $a$ )

Plug  $S$  and  $R$  into form to get  $Q$ :  $\neg\text{Small}(a) \rightarrow \text{LeftOf}(e, a)$ Now plug  $P$  and  $Q$  into  $P \rightarrow Q$ : Cube( $e$ )  $\rightarrow$  ( $\neg\text{Small}(a) \rightarrow \text{LeftOf}(e, a)$ )

# Compound Conditionals

## Summary

### Translation Recipe (For Any Sentence, Not Just Conditionals)

- 1 Recognize the form

**Example** *a is a cube unless it is small*

**The Form:**  $P \text{ unless } Q \rightsquigarrow \neg Q \rightarrow P$

- 2 Translate the parts
  - If the parts themselves contain connectives, apply the recipe to the parts to translate them
- 3 Plug the parts into the form

Let's look at some more examples in Tarski's World

# Translation

## In Class Exercise

*Translate:*

- (19) *If neither Max nor Claire fed Folly at 2:00, then she was hungry*  
 $\neg(\text{Fed}(\text{max}, \text{folly}, 2:00) \vee \text{Fed}(\text{claire}, \text{folly}, 2:00)) \rightarrow \text{Hungry}(\text{folly}, 2:00)$
- (20) *If **b** is a dodecahedron, then if it isn't in front of **d** then it isn't in back of **d** either*  
 $\text{Dodec}(\text{b}) \rightarrow (\neg \text{FrontOf}(\text{b}, \text{d}) \rightarrow \neg \text{BackOf}(\text{b}, \text{d}))$
- (21) *At least one of **a**, **c** and **e** is a cube*  
 $\text{Cube}(\text{a}) \vee \text{Cube}(\text{c}) \vee \text{Cube}(\text{e})$
- (22) ***a** is a tetrahedron unless neither **c** nor **b** are small*  
 $\neg \neg(\text{Small}(\text{c}) \vee \text{Small}(\text{b})) \rightarrow \text{Tet}(\text{a})$
- (23) *If **c** and **d** are both cubes, then one is to the right of the other*  
 $(\text{Cube}(\text{c}) \wedge \text{Cube}(\text{d})) \rightarrow (\text{RightOf}(\text{c}, \text{d}) \vee \text{RightOf}(\text{d}, \text{c}))$

# Translation

## It's Pretty Simple

- *if and only if* translates as  $\leftrightarrow$
- The 'mathematical' use of *just in case* translates as  $\leftrightarrow$
- For Example:

*a is a cube if and only if it is small*

Translates as:

$\text{Cube}(\text{a}) \leftrightarrow \text{Small}(\text{a})$

And:

*e is large just in case d is small*

Translates as:

$\text{Large}(\text{e}) \leftrightarrow \text{Small}(\text{d})$

# Conclusion

## For Today

- We reviewed the meanings of  $\rightarrow$  and  $\leftrightarrow$ :
  - $P \rightarrow Q$  is F if P is T and Q F. Otherwise, it is T.
  - $P \leftrightarrow Q$  is T just in case P and Q have the same truth value. Otherwise, it is F
- We learned how to translate several kinds of English conditionals using  $\rightarrow$  and  $\leftrightarrow$
- We learned a recipe for translating sentences containing connectives
- We saw that the recipe really comes into its own when the connective connects compound sentences