

Formal Proofs and Boolean Logic II

Extending \mathcal{F} with Rules for \neg

William Starr

02.24.09

Outline

- 1 Review
- 2 Formal Rules for \neg
- 3 Using Subproofs
- 4 Proof Strategies
- 5 Conclusion

Announcements

For 02.24

- 1 HW5 is due **now**
- 2 Grades for HW1 & HW3 are posted in the Sakai gradebook
- 3 Do the Homework, it's **really important!**

Review

Proof by Contradiction

- Last class, we learned how to do formal proofs for \vee and \wedge
 - But what about the rules for \neg ?
- That's the topic of Today's class
- Our \neg **Intro** rule will allow us to prove negated claims
- As always, our formal rule mirrors our informal proofs
- So let's review our informal method for proving negated claims

Proof by Contradiction

Proving a Negated Claim

Proof by Contradiction (Official Version)

- 1 To prove that P is false, show that a contradiction \perp follows from P
- 2 To prove that P is true, show that a contradiction \perp follows from $\neg P$

Proving a Negated Claim

To prove $\neg P$, assume P and prove a contradiction \perp

- All contradictions are **impossible**, thus **false**
- If you can show that P leads to a contradiction, then P must be false
- But if P is false, then $\neg P$ must be true

Review

What is a Contradiction Again?

Contradiction

- A **contradiction** is any sentence that cannot possibly be true, or any group of sentences that cannot all be true simultaneously
- The symbol \perp is often used as a short-hand way of saying that a contradiction has been obtained

• Examples:

- 1 $\neg \text{Cube}(a) \wedge \text{Cube}(a)$
- 2 $a = b, b = c, a \neq c$
- 3 $\text{Cube}(a) \wedge \text{Tet}(a)$

Proof by Contradiction

A Simple Example

Claim: This argument is valid

$\neg \text{SameShape}(a, b)$	
$b = c$	
$\neg a = c$	

Proof: We want to show $\neg a = c$ from the premises, so we will use a **proof by contradiction**

- 1 Suppose $a = c$
- 2 Then, from **premise one** $\neg \text{SameShape}(c, b)$ follows by Indiscernibility of Identicals
- 3 But by **premise two**, we know $\text{SameShape}(c, b)$. This is a contradiction, \perp !
- 4 So our **supposition** must have been **false**; that is, $\neg a = c$ must be **true** given the premises

Formal Rules for \neg

Where We Are Going

- The basic idea behind \neg **Intro** is familiar from our informal method of proof by contradiction
 - You can use \neg **Intro** to infer $\neg P$ when you have proven that a contradiction \perp follows from P
- So, to formulate this rule, we must think clearly about what exactly should count as proving \perp
- When have we satisfactorily shown that a contradiction follows from what we are assuming?
- This suggests that we also need a rule handling the introduction of \perp
- The next question is then how to formulate such a rule

Contradictions Again

Varieties Contradiction

- Before looking at \perp **Intro** & \perp **Elim**, we need to recognize a distinction between two kinds of contradictions
 - 1 **Boolean Contradictions**
For example:
 - $\text{Cube}(a), \neg\text{Cube}(a)$
 - $\text{Tet}(a) \wedge \neg\text{Tet}(a)$
 - 2 **Analytic Contradictions**
For example:
 - $\text{Large}(a), \text{Small}(a)$
 - $\text{FrontOf}(a, b), \text{BackOf}(a, b)$
- With these details fresh in our heads, we'll add two rules to \mathcal{F} for handling contradictions

Contradictions

\perp Intro

\perp Intro	
P	
⋮	
$\neg P$	
▶	\perp

- If you've proven a sentence and its negation:
 - You can introduce the \perp symbol to indicate that a contradiction has been derived
- **Important:** This rule only detects **Boolean** contradictions
 - If you had $\text{FrontOf}(a, b)$ and $\neg\text{BackOf}(b, a)$ you could **not** use this rule to introduce \perp
- Fitch is more liberal than \perp **Intro**
 - It allows **Analytic contradictions** to justify introduction of \perp with **Ana Con**

Boolean v. Analytic Contradictions

Within \mathcal{F}

Boolean \perp in \mathcal{F}

1	Cube(a)	
2	$\neg\text{Cube}(a)$	
3	\perp	\perp Intro: 1, 2

- We have P and $\neg P$
- So \perp **Intro** allows us to introduce \perp

Analytic \perp in Fitch

1	Cube(a)	
2	Tet(a)	
3	\perp	Ana Con: 1, 2

- Here we do **not** have P and $\neg P$
- So \perp **Intro** does not give us \perp
- But **Ana Con** does

\perp Elim

What Should \perp Elim Be?

- Remember, all rules come in pairs
- We've stated \perp **Intro**, but we haven't said anything about \perp **Elim**
- What **should** we be able to infer from a contradiction?
- Let's think about it for a minute

Valid Arguments

What If The Premises are Inconsistent?

Logical Consequence, Validity

C is a logical consequence of P_1, \dots, P_n if and only if it is **impossible** for P_1, \dots, P_n to be true while C false

- What follows from a contradiction?
 - Anything!
- Why?
 - It's impossible for it to be true
- So, it is impossible for it to be true while **any** conclusion is false!

Contradictions

\perp Elim

\perp Elim

\perp	
\vdots	
\triangleright P	

- From a contradiction \perp , **any conclusion** follows!
- Why again?

- An inference step is valid just in case it **cannot** lead you from a **true** premise to a false conclusion
- Since the premise \perp in this inference can never be true, the inference can never lead one from a true premise to a false conclusion

Contradictions

Wait, What were We Doing?

- So, now we have two more rules in \mathcal{F} : \perp **Intro**, \perp **Elim**
- But wait, why did we start talking about \perp in the first place?
- Because contradictions are essential to formulating our \neg **Intro** rule
 - Recall that proving $\neg P$ involves assuming P and showing that a contradiction follows from this assumption
- So now we'll learn \neg **Intro**

\neg Intro

From Informal to Formal Proof

Proving a Negative Claim

- To prove $\neg P$, assume P and prove a contradiction using this assumption
- This is an example of **Proof by Contradiction**

Example Informal Proof

From $a = b$ and $b \neq c$ we will prove $a \neq c$.
 We use proof by contradiction. **Proof:**
 Suppose $a = c$. Well, $b = c$ follows from this assumption and premise one by Ind. of Id.'s. But, this contradicts premise two, \perp . So our assumption was wrong, in which case $a \neq c$.

\neg Intro

P	
\vdots	
\perp	
\triangleright $\neg P$	

To **prove** $\neg P$:

- 1 Assume P
- 2 Derive \perp (using \perp **Intro**)
- 3 Conclude $\neg P$
(Discharging assumption of P)

\neg Intro

An Example

1	a = b	
2	b \neq c	
3	a = c	
4	b = c	= Elim : 3, 1
5	\perp	\perp Intro : 2, 4
6	a \neq c	\neg Intro : 3-5

✓ **Goal**: a \neq c

\neg Intro	
	P
	⋮
	\perp
▷	\neg P

\perp Intro	
	P
	⋮
	\neg P
▷	\perp

Some More Examples

- Let's do a formal proof for 6.25:

$$\frac{\neg A \wedge \neg B}{\neg(A \vee B)}$$

- Let's also finish the proof from slide 26 of 02.19
 - This will use \perp **Elim**

\neg Intro

Another Example

ARGUMENT 1: Analytic \perp Revisited

1	\neg SameShape(a, b)	
2	b = c	
3	a = c	
4	\neg SameShape(c, b)	= Elim : 1,3
5	\perp	Ana Con : 2, 4
6	a \neq c	\neg Intro : 3-5

Goal: a \neq c

Informal Proof

We want to show a \neq c, so we use proof by contradiction. **Proof**: Suppose a = c. From premise one it follows that \neg SameShape(c, b), by Ind. of Id. But this contradicts premise two which requires that c is b. So our assumption (a = c) was wrong, hence a \neq c follows.

\neg Intro	
	P
	⋮
	\perp
▷	\neg P

- There is no rule in \mathcal{F} which justifies line 5
- But this is what we need to prove a \neq c!
- So, this proof can't be finished in \mathcal{F}
- We *can* finish it in **Fitch!**

Negation

\neg Elim

\neg Elim	
	$\neg\neg$ P
	⋮
	\perp
▷	P

Simple Example		
1	$\neg\neg$ Cube(a)	
2	Cube(a)	\neg Elim : 1

- If $\neg\neg$ P is true, so is P
- \neg **Intro** allows us to use proof by contradiction to prove P
 - We simply use \neg **Intro** to prove $\neg\neg$ P and then apply \neg **Elim**
- This may seem like an obvious and useless rule, but it is in fact quite useful

\neg Elim

An Example

ARGUMENT 2

$$\begin{array}{|l} \text{Tet}(e) \vee \text{Cube}(a) \\ \neg\text{Tet}(e) \\ \hline \text{Cube}(a) \end{array}$$

Informal Proof of Argument 2

We will use a proof by contradiction. Suppose $\neg\text{Cube}(a)$. This pretty clearly contradicts the first premise. To be sure, we'll take it in cases. In the first case ($\text{Tet}(e)$) the contradiction is clear. In the second case ($\neg\text{Cube}(a)$) we also have a contradiction. So our assumption must have been wrong. Hence, $\text{Cube}(a)$ must be true given the premises.

Let's make this into a formal proof in Fitch

Subproofs

The Big Picture

- Subproofs correspond to things we do in informal proofs:
 - They correspond to the **cases** of a proof by cases
 - They correspond to the temporary assumption in a proof by contradiction
- Just like cases and temporary assumptions, there are certain important **restrictions** on subproofs

Cases

The Constraints

ARGUMENT 3

$$\begin{array}{|l} (\text{Cube}(c) \wedge \text{Small}(c)) \vee (\text{Tet}(c) \wedge \text{Small}(c)) \\ \hline \text{Small}(c) \wedge \text{Cube}(c) \wedge \text{Tet}(c) \end{array}$$

Pseudo-Proof of Argument 3

We will use a proof by cases based on premise one. **Case 1:** Suppose $(\text{Cube}(c) \wedge \text{Small}(c))$. Then $\text{Small}(c)$ follows. **Case 2:** Suppose $\text{Tet}(c) \wedge \text{Small}(c)$. Then $\text{Small}(c)$ follows. So, $\text{Small}(c)$ follows in either case. But in case 1 we had $\text{Cube}(c)$ and in case 2 we had $\text{Tet}(c)$, hence our conclusion follows: $\text{Small}(c) \wedge \text{Cube}(c) \wedge \text{Tet}(c)$.

- Why **pseudo-proof**?
 - Argument 3 is **not valid**
 - This 'proof' leads us from a **possible premise** to an **impossible conclusion**
 - That's exactly what proofs **aren't supposed to do**

Cases

The Constraint

Pseudo-Proof of Argument 3

We will use a proof by cases based on premise one. **Case 1:** Suppose $(\text{Cube}(c) \wedge \text{Small}(c))$. Then $\text{Small}(c)$ follows. **Case 2:** Suppose $\text{Tet}(c) \wedge \text{Small}(c)$. Then $\text{Small}(c)$ follows. So, $\text{Small}(c)$ follows in either case. **But in case 1 we had $\text{Cube}(c)$ and in case 2 we had $\text{Tet}(c)$** , hence our conclusion follows: $\text{Small}(c) \wedge \text{Cube}(c) \wedge \text{Tet}(c)$.

- Where exactly does this proof go wrong?
 - When we reached back into cases that were already finished
 - The assumptions and conclusions of a case are only live within that case, once you are done, you are done
 - What happens in a case, stays in a case
 - Although we can cite the case as a whole, as in proof by cases, we can't pick claims out of it to use later in the proof

Temporary Assumptions

The Constraints

- In proof by contradiction, like in proof by cases, we make a temporary assumption:
 - We assume P and try to show \perp
 - But P is a **temporary assumption**
 - And anything we infer from it is also **temporary**
 - Once we show \perp , we **discharge** the assumption of P
 - This temporary assumption of P and the things we infer from it corresponds to a subproof
 - Once this assumption is **discharged**, we can't reach back into the subproof, only cite the subproof as a whole

Subproofs

Drawing the Connections

Proof with A Subproof

```

⋮
|
| A
|-----
| ⋮
| B
|
| ⋮
|
| C
  
```

- A subproof involves a **temporary assumption**
 - Just like proof by contradiction
 - Just like proof by cases
- And just like the temporary assumptions of our informal proof methods:
 - Once the temporary assumption has been **discharged**, you cannot use any of the individual lines of the subproof to justify additional steps

Subproofs

Guidelines for Use

Guidelines for Using Subproofs

- 1 Once a subproof has ended, you can **never** cite one of its lines individually for any purpose, although you may cite the subproof as a whole (as in \vee **Elim** & \neg **Intro**)
- 2 In justifying a step of a proof, you may cite any earlier line of the **main proof**, or any subproof that **has not ended**

Let's do exercise 6.17 to solidify these points

Proof Strategies

How to Approach a Formal Proof

- 1 Understand what the sentences are saying
- 2 Decide whether you think the conclusion follows from the premises
- 3 If you don't think so, try to find a counterexample
- 4 If you do think so, try to give an informal proof
- 5 Use this informal proof to guide your formal proof
- 6 If you get stuck try working backwards

Summary

Negation

- We learned two new negation rules: \neg **Intro**, \neg **Elim**
- \neg **Intro** mirrors the proof by contradiction method
- To mimic this method in \mathcal{F} we introduced the \perp symbol and two rules for it: \perp **Intro**, \perp **Elim**
- Proof by contradiction isn't just good for proving negated claims
 - It can also be used to prove positive claims

Summary

Subproofs & Strategy

- Mastering \mathcal{F} involves mastering subproofs
- Just like cases and reductio assumptions, there are constraints on how you can use subproofs
- We learned these constraints and the perils they guard us against
- We also learned how to approach proofs
 - There's strategy to it!
 - **Don't** just try to shuffle symbols!