Methods of Proof for Boolean Logic

Proof by Contradiction

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Announcements

For 02.17

1. HW4 is due now
2. HW1-3 will be returned on Thursday

Outline

Proof by Contradiction

Proving a Negated Claim

- Okay, we've done proofs for conjunction and disjunction, but what about negation?
- How would you go about proving a negated claim like ¬Cube(a)?
- Well, ¬Cube(a) is true if and only if Cube(a) is false
- So, we can prove ¬Cube(a) by showing that Cube(a) is false
- But how do we do that?

Proof by Contradiction
Proving Something False

- There is a very important method for proving something false
  - Proof by Contradiction
  - A.k.a **Indirect Proof**, **Reductio ad Absurdum**

The Basic Idea of Proof by Contradiction

To show that P is false, it suffices to show that something which cannot possible be true, i.e. a contradiction, follows from P

Proof by Contradiction
What is a *Contradiction*?

Contradiction

- Intuitively, a contradiction is any sentence that cannot possibly be true, or any group of sentences that cannot all be true **simultaneously**
- The symbol $\perp$ is often used as a short-hand way of saying that a contradiction has been obtained

- Examples:
  1. $\neg\mathsf{Cube}(a) \land \mathsf{Cube}(a)$
  2. $a = b, b = c, a \neq c$

Proof by Contradiction
What is It?

Proof by Contradiction (Preliminary Version)

To prove that P is false, show that a contradiction $\perp$ follows from P

Proving a Negated Claim

To prove $\neg$P, assume P and prove a contradiction $\perp$

- Contradictions are impossible, so false
- If you can show that P leads to a contradiction, then P must be false
- But if P is false, then $\neg$P must be true

Proof by Contradiction
A Simple Example

Claim: This argument is valid

$$\begin{array}{|l} \neg\mathsf{SameShape}(a, b) \\ b = c \\ \hline \neg a = c \end{array}$$

Proof: We want to show $\neg a = c$ from the premises, so we will use a proof by contradiction

1. Suppose $a = c$
2. Then, from premise one $\neg\mathsf{SameShape}(c, b)$ follows by Indiscernibility of Identicals
3. But by premise two, we know $\mathsf{SameShape}(c, b)$. This is a contradiction, $\perp$!
4. So our supposition must have been false; that is, $\neg a = c$ must be true given the premises

Proof by Contradiction
In Class Exercise

Write an informal proof of this argument. Do a proof by contradiction.

$$
\begin{array}{|l}
\mathsf{Cube}(b) \lor \mathsf{LeftOf}(b,c) \\
\mathsf{SameCol}(b,c) \\
\hline
\neg\mathsf{Tet}(b)
\end{array}
$$

Proof by Contradiction
Official Version

Proof by Contradiction (Official Version)

1. To prove that P is false, show that a contradiction $\bot$ follows from P
2. To prove that P is true, show that a contradiction $\bot$ follows from $\neg$P

- Proof by contradiction can also be used for proving un-negated claims

Proof by Contradiction
Used to Prove an Un-negated Claim

$$
\begin{array}{|l}
b = c \land \mathsf{SameShape}(c,a) \\
\mathsf{Cube}(b) \\
\hline
\mathsf{Cube}(a)
\end{array}
$$

Proof: We will use a proof by contradiction. First, suppose the conclusion is false, i.e. $\neg\mathsf{Cube}(a)$. By premise 1 we know that $\mathsf{SameShape}(c,a)$, so it must be that $\neg\mathsf{Cube}(c)$. But premise 1 tells us that $b = c$ and together with premise 2 this entails that $\mathsf{Cube}(c)$, by the Indiscernibility of Identicals. This contradicts our early conclusion, so our supposition must have been false. Thus, the conclusion must be true when the premises are.

Proof by Contradiction
In Class Exercise

Write an informal proof of this argument, phrased in complete, well-formed English sentences. ***Hint****: try a proof by contradiction. I encourage you to work in groups.*

5.15

$$
\begin{array}{|l}
\mathsf{Tet}(b) \\
\mathsf{Cube}(c) \\
\mathsf{Larger}(c,b) \lor c = b \\
\hline
\mathsf{Smaller}(b,c)
\end{array}
$$

Proof by Contradiction

A More Advanced Example

Proof

We will do a proof by cases based on premise three:

1. Suppose $\neg$Small(d). We want to show $\mathsf{d} = \mathsf{c} \vee \mathsf{d} = \mathsf{b}$, let's do this by indirect proof. This requires deriving $\perp$ from the additional supposition that $\neg(\mathsf{d} = \mathsf{c} \vee \mathsf{d} = \mathsf{b})$. First, note that by DeMorgan's Laws this implies $\mathsf{d} \neq \mathsf{c}$ and $\mathsf{d} \neq \mathsf{b}$. From this and our first supposition, premise two clearly requires that $\neg$Dodec(d). But, from premise four and our original supposition, premise one clearly requires that Dodec(d). These requirements are contradictory, $\perp$.
2. Now suppose $\mathsf{d} = \mathsf{b}$. Then $\mathsf{d} = \mathsf{c} \vee \mathsf{d} = \mathsf{b}$ follows immediately by disjunction intro

In either case $\mathsf{d} = \mathsf{c} \vee \mathsf{d} = \mathsf{b}$ follows, so the argument is valid

Argument 4

Dodec(d) $\vee$ Tet(d) $\vee$ Small(d)

$\neg$Dodec(d) $\vee$ (d = c) $\vee$ Small(d)

$\neg$Small(d) $\vee$ d = b

$\neg$Tet(d)

$\mathsf{d} = \mathsf{c} \vee \mathsf{d} = \mathsf{b}$

- We had a proof by contradiction inside a proof by cases!
- This is like exercise 5.17!

Valid Arguments

What If The Premises are Inconsistent?

Logical Consequence, Validity

C is a logical consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$ if and only if it is impossible for $\mathsf{P}_1, \ldots, \mathsf{P}_n$ to be true while C false

- Now think about an argument with inconsistent premises
  - Is it valid?
- Yes!
  - Why: it's impossible for the premises to be true
- So, it is impossible for the premises to be true while the conclusion is false!
- But, crucially, the argument is not sound

Summary

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- We learned a powerful proof method:
  - Proof by contradiction
- We wrapped our head around the fact that any argument with contradictory premises is valid
  - Importantly, though, such an argument is never sound