

Methods of Proof for Boolean Logic

William Starr

02.12.09

Outline

- 1 Introduction
- 2 Valid Inference Steps
- 3 Proof by Cases

Announcements

For 02.12

- 1 HW 4 due next Monday (02.17)

The Big Picture

Where is Today?

- You are taking a logic class
- Logic is mainly about *logical consequence*
 - It's about conclusions following (or not following) from premises
- So far, we've explored two methods for understanding logical consequence:
 - 1 **Proof** (Ch.2: informal, formal)
 - 2 **Tautological Consequence** (Ch.4: truth tables)
- However, we have not discussed how the methods of **proof** can be used for the Booleans
- **That will be our project for today**

The Big Picture

But Wait...

- We had to learn truth tables, why proofs too?
- Truth tables provide powerful logical tools for the Booleans, but have two significant limitations:
 - 1 **Impractical:** Truth tables get extremely large as they get more interesting. An interesting scientific argument could have well over 14 atomic sentences. That would take a table with over 16,000 rows!
 - 2 **Limited Applications:** As we learned yesterday, there are logical consequences that **aren't** tautological consequences. That's because truth tables are blind to the logic of expressions other than the Booleans, such as $=$.
- The methods of **proof** fill this gap admirably

Proofs

The Way Forward

- We've studied the basics of proofs but we haven't done any proofs involving the Boolean connectives
- Today we'll be discussing the **informal** methods of proof for the Booleans
- i+-i Next week we'll extend our formal proof system (\mathcal{F}) with formal rules for the Boolean connectives
 - These formal rules will closely mirror the informal proof methods discussed today

Proof

What is it?

Proof

A **proof** is a step-by-step demonstration which shows that a conclusion C must be true in any circumstance where some premises, say P_1 and P_2 are true

- 1 The step-by-step demonstration of C can proceed through **intermediate conclusions**
- 2 It may not be obvious how to show C from P_1 and P_2 , but it may be obvious how to show C from some other claim Q that is an obvious consequence of P_1 and P_2
- 3 Each step must provide incontrovertible evidence for the next

Proof

Steps, What?

The Nature of Steps

Each step of a proof involves an appeal to certain facts about the **meaning** of the vocabulary involved. These facts are what we implicitly appeal to when we say a step is *obvious*

- What kind of facts?
- Facts which guarantee that the step will never lead us from something **true** to something **false**
- Let's consider an old example

Proof

An Old Inference Step

Indiscernibility of Identicals

If n is m , then whatever is true of n is also true of m
(where ' n ' and ' m ' are names)

- This is a fact about the meaning of *is*
- This fact guarantees that if it is true that n is m , then it is true that whatever holds of n also holds of m
- In other words it could never lead us from true claims to false ones
- This is the essence of a valid inference step

Proof

New Steps

- So we need to think about which inference steps negation, conjunction and disjunction support
 - That is, we need to think about what they **mean**
 - We've already started doing this:
 - 1 \wedge takes the 'worst' truth value
 - 2 \vee takes the 'best' truth value
 - 3 \neg flips the truth value
- Now we just need to think about what these facts imply in terms of inference steps and proof methods

Conjunction

Elimination

- Suppose you have proved a conjunction $P \wedge Q$
- From looking at the truth table for \wedge or thinking about the meaning of *and*, both P and Q are clearly consequences of $P \wedge Q$
- This inference pattern is called **conjunction elimination**

Truth Table for \wedge

P	Q	$P \wedge Q$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Conjunction Elimination

- 1 From $P \wedge Q$ you can infer P
- 2 From $P \wedge Q$ you can infer Q

Conjunction

Elimination

- Conjunction elimination is pretty obvious:
 - 1 Jay walks and Kay talks
 - So Jay walks
 - 2 $\text{Large}(a) \wedge \text{Cube}(a)$
 - So $\text{Cube}(a)$
- This inference is so obvious that we rarely take the time to mention that we are making it
- In your informal proofs you don't have to mention it either, but you should be aware that you are making it and why it's a valid inference step

Conjunction

Introduction

- There's a similar inference step for inferring a conjunction from its conjuncts:

Conjunction Introduction

If you have proven both P and Q , you can infer $P \wedge Q$

- Again, this is so obvious that we never mention such a step
- You don't have mention it in your proofs, but you should understand it

Disjunction

Introduction

- Suppose you have proven P
- Then you can infer $P \vee Q$, **no matter what Q is**
- Why?

Truth Table for \vee

P	Q	$P \vee Q$
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

- If P is true, then $P \vee Q$ is true, regardless of what Q 's truth value is!
- This is a valid inference step, since it is guaranteed to lead us from true claims to true claims

Disjunction

Introduction

So \vee supports this inference step:

Disjunction Introduction

- 1 From A you can infer $A \vee B$
- 2 From A you can infer $B \vee A$

- It may seem useless, to infer from:
 - (1) Likes(jay, circles)
 that
 - (2) Likes(jay, circles) \vee Likes(kay, squares)
- But, we will find uses for these kinds of inferences

Informal Proof

A Rule of Thumb

Rule of Thumb for Informal Proofs

In an informal proof, it is always legitimate to move from P to Q if both you & your audience already know that Q is a logical consequence of P

- We've all learned the following equivalences:

DeMorgan's Laws

- 1 $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
- 2 $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

Double Negation

$$\neg\neg P \Leftrightarrow P$$

- So in an informal proofs for this class you could say: "From $\neg(\text{Cube}(a) \wedge \text{Tet}(b))$ it follows by DeMorgan's Laws that $\neg\text{Cube}(a) \vee \neg\text{Tet}(b)$..."

Informal Proof

A Rule of Thumb

- Of course, if you are asked to prove one of DeMorgan's Laws, or you are talking logic with stranger you shouldn't appeal to DeMorgan's Laws
- So we have three new inference steps, some equivalences and a rule of thumb
- So what?
- Well, now we can prove some stuff

An Example Proof

Argument 1

$$\begin{array}{l} \neg(A \vee B) \\ \hline \neg A \end{array}$$

- Let's give an informal proof of this inference

Proof of Argument 1

We are given $\neg(A \vee B)$, which is equivalent to $\neg A \wedge \neg B$ by DeMorgan's Law (2). So $\neg A$ follows immediately.

- We used Conjunction Elimination in this last step, but there was no need to explicitly say so

Another Example Proof

Inference 2

Argument 2

$$\begin{array}{l} a = b \wedge \neg \text{Cube}(a) \\ \hline \neg(\text{Cube}(a) \vee \text{Cube}(b)) \end{array}$$

- Let's give an informal proof of this inference too

Proof of Argument 2

We are given that $a = b$ and $\neg \text{Cube}(a)$, so by the indiscernibility of identicals $\neg \text{Cube}(b)$. Now we have $\neg \text{Cube}(a) \wedge \neg \text{Cube}(b)$, but by DeMorgan's Law (2) this is equivalent to $\neg(\text{Cube}(a) \vee \text{Cube}(b))$, which is our desired conclusion.

Moving On

Proof Methods

- We have discussed:
 - 1 Conjunction Intro and Elim
 - 2 Disjunction Intro
- But what about:
 - 1 Disjunction Elim
 - 2 Negation!
- As it turns out, these are not formulated as simple rules, but as more structured **methods of proof**

Proof by Cases

The Basics

- The first method of proof we are going to learn about is called **proof by cases**
- In short, it will allow us to use disjunctions to prove things
- Let's first look at an example where a disjunction is used to prove something

Proof by Cases

An Example

Disjunctions in a Proof

Apollo went to go buy a smoothie at either the George St. Co-op or Whole Foods. He always buys the cheapest smoothie. Right now, the cheapest smoothie at both the George St. Co-op and Whole Foods is Odwalla. So, **if he went to George St. he will buy Odwalla and if he went to Whole Foods he will also buy Odwalla. So either way he'll come back with an Odwalla smoothie.**

- Our **first premise** was a **disjunction**
- Here's how we used it:
 - We reasoned that **if the first disjunct was true, Apollo would come back w/Odwalla**
 - We then reasoned that **if the second was true, Apollo would come back w/Odwalla**
 - We concluded that Apollo would come back w/Odwalla
- Why? Because the meaning of *or* guarantees that at least one disjunct of the first premise is true

Proof by Cases

The Method

So, our strategy was:

- We had $A \vee B$ and wanted to show C
- So, we showed that if A was true C was true
- **And** that if B was true, C was true
- This is a valid proof of C from $A \vee B$, since:
 - $A \vee B$ guarantees that either A is true or B is true
 - And, we've shown that either way C is true
- This is called **proof by cases**, since it breaks the proof into a number of cases, one for each disjunct

Proof by Cases

Official Version

So, more abstractly our strategy was this:

Proof by Cases (Disjunction Elimination)

To prove C from $P_1 \vee \dots \vee P_n$ using this method, show C from each of P_1, \dots, P_n

- From our disjunctive premise we know at least one disjunct is true
- So showing that the truth of any one of them guarantees the truth to C , suffices to show that C follows from our disjunctive premise

Proof by Cases

An Example

Argument 3

$$(Cube(c) \wedge Small(c)) \vee (Tet(c) \wedge Small(c))$$

$$Small(c)$$

- Let's give an informal proof of this inference

Proof of Argument 3

We have a **disjunctive premise**, so we will use the proof by cases method. We have two disjuncts in our premise so we'll break our proof into two cases:

- Suppose $Cube(c) \wedge Small(c)$. Then $Small(c)$ follows!
- Suppose $Tet(c) \wedge Small(c)$. Then again, $Small(c)$!

Either way $Small(c)$ follows

Proof by Cases

Another Example

Claim: the following argument is valid

$$Cube(a) \vee Smaller(a, b)$$

$$\neg Cube(a) \vee Smaller(a, c)$$

$$Smaller(b, c)$$

$$Smaller(a, c)$$

Proof: Given the first premise, we'll try a proof by cases:

- Suppose $Cube(a)$. By the **second premise** we know that either $Cube(a)$ is false or $Smaller(a, c)$. By assumption, $Cube(a)$ is true. So, it must be the case that $Smaller(a, c)$
- Suppose $Smaller(a, b)$. We are given that $Smaller(b, c)$ and $Smaller(,)$ is transitive, so $Smaller(a, c)$

We've shown that in either case $Smaller(a, c)$ follows

Proof by Cases

In Class Exercise

Write an informal proof of this argument, phrased in complete, well-formed English sentences. **Hint:** try a proof by cases. I encourage you to work in groups.

$$Smaller(a, c) \vee FrontOf(a, b)$$

$$Larger(a, c) \vee BackOf(b, a)$$

$$Between(c, a, b)$$

$$FrontOf(a, b)$$

Summary

02.12

- We learned three valid inference steps for the Booleans:
 - Conjunction Intro/Elim and Disjunction Intro
 - You don't have to mention the conjunction steps
- We learned some tricks for informal proofs:
 - You can usually appeal to things like DeMorgan's Laws
 - You can appeal to facts about the vocabulary, e.g. *transitivity*, *inverseness*, at least one disjunct has to be true etc.
- We learned a powerful proof method:
 - Proof by cases