

# The Logic of Boolean Connectives

## Tautological Equivalence & Consequence

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## Outline

- 1 Review
- 2 Equivalence
- 3 Consequence

## Announcements

For 02.10

- 1 HW3 is due **now**

## Review

### The Boolean Connectives

#### Truth Table for $\neg$

P	$\neg P$
TRUE	FALSE
FALSE	TRUE

#### Truth Table for $\wedge$

P	Q	$P \wedge Q$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

#### Truth Table for $\vee$

P	Q	$P \vee Q$
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

- $\neg$  flips the value
- $\wedge$  takes the 'worst' value
- $\vee$  takes the 'best' value

## The Basics

## Step 1: The Reference Columns

- We are going to construct a truth table for  
(1)  $\text{Cube}(a) \vee \neg\text{Cube}(a)$

First, some columns:

- A column for each atomic sub-sentence of (1), called a **reference column**

Truth Table for (1)	
Cube(a)	Cube(a) $\vee$ $\neg\text{Cube}(a)$
T	
F	

- A column for (1) itself
  - Now, we fill the reference column w/truth values
  - One **row** for each **unique logical possibility**

## Truth Table for (1)

Cube(a)	Cube(a) $\vee$ $\neg\text{Cube}(a)$
T	F
F	T

- Next, we fill in the column beneath the **innermost connective**  $\neg$ :
  - In the first row,  $\text{Cube}(a)$  is T so  $\neg\text{Cube}(a)$  is F
  - In the second row,  $\text{Cube}(a)$  is F so  $\neg\text{Cube}(a)$  is T

## The Basics

## Step 3: The Main Connective

## Truth Table for (1)

Cube(a)	Cube(a) $\vee$ $\neg\text{Cube}(a)$
T	T F
F	T T

- Lastly, we fill in the columns beneath the **outermost connective**  $\vee$ :
  - In the first row,  $\text{Cube}(a)$  is T and  $\neg\text{Cube}(a)$  is F, so their disjunction is T
  - In the first row,  $\text{Cube}(a)$  is F and  $\neg\text{Cube}(a)$  is T, so their disjunction is T
- This column** lists every logically possible truth value for (1)

## The Basics

## Summary

## How to Construct a Truth Table for P

- Reference Columns:** Draw a column for each atomic sub-sentence of P, these columns are called the **reference columns** and are filled with every possible combination of truth-values for the sub-sentences
- Inside Out:** Draw a column for P itself. Then fill in the column below P's **innermost connective**. Repeat for the next innermost connective, until you get to the main connective.
- Main Connective:** Fill in the column under the **main connective**. This row lists the possible truth values of P

# Equivalence

## Two Varieties

### Logical Equivalence

Two sentences are **logically equivalent** if and only if they have the same truth values in every possible situation

- For example:  $\text{Tet}(a)$  and  $\neg\neg\text{Tet}(a)$  are logically equivalent

### Tautological Equivalence

Two sentences are **tautologically equivalent** just in case the columns under their main connectives in a **joint truth table** are identical

- What is a **joint truth table**?

# Equivalence

## Joint Truth Tables

- The idea of a **joint truth table** is quite simple, just add a column on the right for another formula and calculate as before

### Joint Truth Table

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

- 1 Ref. columns
- 2 Inner connectives
- 3 Main connectives

- The columns under the main connectives are **identical**
- So, these two sentences are **tautologically equivalent**

# Equivalence

## Logical vs. Tautological

- We started by characterizing two kinds of equivalence
  - 1 Logical
  - 2 Tautological
- Every two sentences that are tautologically equivalent are logically equivalent
- Does the reverse hold?
- No, this pair is logically equivalent:
  - (2)  $a = b \wedge \text{Cube}(a)$
  - (3)  $a = b \wedge \text{Cube}(b)$
- But we can show with Boole that they **aren't tautologically equivalent**

# Consequence

## Two Varieties Again

### Logical Consequence

C is a **logical consequence** of  $P_1, \dots, P_n$  just in case it is **logically impossible** for C to be false while  $P_1, \dots, P_n$  are true

- We've already met this concept of validity/consequence
- It doesn't help us much with **figuring out** whether an argument is valid
- Proof provides one method, truth tables another:

### Tautological Consequence

C is a **tautological consequence** of  $P_1, \dots, P_n$  just in case every row in their **joint truth table** that lists T under  $P_1, \dots, P_n$  also lists T under C

## Tautological Consequence

An Example

## Argument 1

$$\begin{array}{l} A \vee B \\ \neg A \\ \hline B \end{array}$$

## Joint Truth Table for Argument 1

A	B	$A \vee B$	$\neg A$	B
T	T	T	F	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

- First two columns for the premises
- Last column for conclusion
- Every row where both premises are T, the conclusion is T
- So B is a **tautological consequence** of  $A \vee B$  and  $\neg A$

## Tautological Consequence

Another Example: Exercise 4.20

Let's run through exercise 4.20

## Tautological Consequence

In-Class Exercise

## Exercise 4.21

$$\begin{array}{l} \text{Taller}(\text{claire}, \text{max}) \vee \text{Taller}(\text{max}, \text{claire}) \\ \text{Taller}(\text{claire}, \text{max}) \\ \hline \neg \text{Taller}(\text{max}, \text{claire}) \end{array}$$

## Consequence

Questions

- 1 If C is a tautological consequence of  $P_1, \dots, P_n$ , is C a logical consequence of  $P_1, \dots, P_n$ ?
  - Yes, clearly
- 2 If C is a logical consequence of  $P_1, \dots, P_n$ , is C a tautological consequence of  $P_1, \dots, P_n$ ?
  - No, let's show it using Boole to construct a joint truth table for this argument:

## Argument 2

$$\begin{array}{l} a = b \wedge b = c \\ \hline a = c \end{array}$$

# Taut Con

Tautological Consequence in Fitch

- Truth tables provide a powerful but purely mechanical procedure to test for logical consequence
- But, they often get really tedious and long
- But, that's what computers are good at
- Fitch has a built-in mechanism for testing for tautological consequence
  - **Taut Con**
- Much like **Ana Con**, this is not a rule of inference, but a computational mechanism
- Let's run through exercise 4.26

# Summary

What we did today

- We reviewed the construction of truth tables, by hand & with Boole
- We applied truth tables to the problem of precisely defining:
  - ① Logical equivalence
  - ② Logical consequence
- We came up with two similar concepts:
  - ① Tautological equivalence
  - ② Tautological consequence
- In each case Tautological implied Logical, but Logical did **not** imply Tautological