

The Logic of Boolean Connectives

Truth Tables, Tautologies & Logical Truths

William Starr

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Outline

- 1 Introduction & Review
- 2 Truth Tables
- 3 Logical Truths & Tautologies

Announcements

For 02.05.09

- 1 HW3 is due next Tuesday

Introduction

Truth Functions

- Last class, we learned the meaning of \wedge, \vee, \neg in terms of **truth functions**
- We also saw that truth functions allowed us to do something useful:
 - Figure out the truth value of a complex sentence from the truth values of its atomic parts, and vice versa
 - For example, we know that $\neg(\text{Cube}(a) \vee \text{Cube}(b))$ is true in a world where neither a nor b are cubes, since:
 - If $\text{Cube}(a)$ and $\text{Cube}(b)$ are false then $\text{Cube}(a) \vee \text{Cube}(b)$ is false
 - If $\text{Cube}(a) \vee \text{Cube}(b)$ is false, then $\neg(\text{Cube}(a) \vee \text{Cube}(b))$ is true

Introduction

Truth Tables

- These calculations — ones like the one we just went through — are a bit clunky
- **Truth tables** provide a much cleaner method
- As it turns out, constructing truth tables will also provide us with a helpful way to understand 3 core logical concepts:
 - 1 Logical Consequence
 - 2 Logical Truth
 - 3 Logical Equivalence
- Today, we'll learn all about truth tables & these applications!

Review

The Boolean Connectives

Truth Table for \neg

P	$\neg P$
TRUE	FALSE
FALSE	TRUE

Truth Table for \wedge

P	Q	$P \wedge Q$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Truth Table for \vee

P	Q	$P \vee Q$
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

- \neg flips the value
- \wedge takes the 'worst' value
- \vee takes the 'best' value

The Basics

Step 1: The Reference Columns

- We are going to construct a truth table for
(1) $\text{Cube}(a) \vee \neg \text{Cube}(a)$

First, some columns:

- 1 A column for each atomic sub-sentence of (1), called a **reference column**
- 2 A column for (1) itself
 - Now, we fill the reference column w/truth values
 - One **row** for each **unique logical possibility**

Truth Table for (1)

$\text{Cube}(a)$	$\text{Cube}(a) \vee \neg \text{Cube}(a)$
T	
F	

The Basics

Step 2: Inside Out

Truth Table for (1)

$\text{Cube}(a)$	$\text{Cube}(a) \vee \neg \text{Cube}(a)$
T	F
F	T

- Next, we fill in the column beneath the **innermost connective** \neg :
 - In the first row, $\text{Cube}(a)$ is T so $\neg \text{Cube}(a)$ is F
 - In the second row, $\text{Cube}(a)$ is F so $\neg \text{Cube}(a)$ is T

The Basics

Step 3: The Main Connective

Truth Table for (1)

Cube(a)	\vee	\neg Cube(a)
T		T F
F		T T

- Lastly, we fill in the columns beneath the **outermost connective** \vee :
 - In the first row, Cube(a) is T and \neg Cube(a) is F, so their disjunction is T
 - In the first row, Cube(a) is F and \neg Cube(a) is T, so their disjunction is T
- **This column** lists every logically possible truth value for (1)

The Basics

Summary

How to Construct a Truth Table for P

- 1 **Reference Columns:** Draw a column for each atomic sub-sentence of P, these columns are called the **reference columns** and are filled with every possible combination of truth-values for the sub-sentences
- 2 **Inside Out:** Draw a column for P itself. Then fill in the column below P's **innermost connective**. Repeat for the next innermost connective, until you get to the main connective.
- 3 **Main Connective:** Fill in the column under the **main connective**. This row lists the possible truth values of P

Reference Columns

There is More to It...

- When you have more than one atomic sub-sentence, filling in the reference columns requires more thought
- Remember that each **row** of the reference columns lists **a unique logical possibility**
- Also remember that there is supposed to be a row for **every unique possibility**
- Okay, well how many rows would we need for a formula with 2 atomic sub-sentences?

Reference Columns

How Many Rows?

Let's figure it out:

- We have 2 atomic sub-sentences
- Each can have 2 different truth values (T,F)
- So there $2^2 = 4$ possible combinations of truth values and atomic sub-sentences
- Therefore, a table for a formula with 2 atomic sub-sentences needs 4 rows

You Always Need 2^n Rows

In general, if there are n atomic sub-sentences of P then there will be 2^n possible assignments of truth values to those atomic sub-sentences, in which case the truth table for P should have 2^n rows.

Another Example

Step 1

- Let's construct a truth table for:
(2) $\neg(\text{Cube}(a) \wedge \text{Cube}(b))$
- We need 2 reference columns:
 - In them, we need a row for each of the 4 logical possibilities
 - Cube(a) can be T and Cube(b) T; Cube(a) T and Cube(b) F, and so on.

Truth Table for (2)

Cube(a)	Cube(b)	$\neg(\text{Cube}(a) \wedge \neg\text{Cube}(b))$
T	T	
T	F	
F	T	
F	F	

Another Example

Steps 2 & 3

Truth Table for (2)

Cube(a)	Cube(b)	$\neg(\text{Cube}(a) \wedge \neg\text{Cube}(b))$
T	T	T F F
T	F	F T T
F	T	T F F
F	F	T F T

- Next, the innermost connective \neg :
 - It will flip each value of Cube(b)
- Now, the next innermost \wedge :
 - \wedge takes worst value of the pair
- Finally, the main connective \neg :
 - \neg flips the value of the conjunction we just computed

Building Reference Columns

A Helpful Routine

- You need to list each possibility exactly once when filling in reference columns
- Here's a helpful routine for this:
 - In the innermost ref. column, you alternate T's and F's
 - In the next innermost column, you double that alternation, and so on for any more rows

Truth Table for (2)

Cube(a)	Cube(b)	$\neg(\text{Cube}(a) \wedge \neg\text{Cube}(a))$
T	T	
T	F	
F	T	
F	F	

Let's put this routine to work

Yet Another Example

Step 1

We'll construct a table for:

(3) $(\text{Cube}(a) \wedge \neg\text{Cube}(b)) \vee \neg\text{Cube}(c)$

Let A = Cube(a), B = Cube(b), C = Cube(c)

Table for (3)

A	B	C	$(A \wedge \neg B) \vee \neg C$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

- First, the 8 rows of the reference columns:
 - Alternate on innermost column
 - Double this alternation on the next
 - Double again

Yet Another Example

Steps 2 & 3

Table for (3)

A	B	C	$(A \wedge \neg B) \vee \neg C$
T	T	T	F F F F
T	T	F	F F T T
T	F	T	T T T F
T	F	F	T T T T
F	T	T	F F F F
F	T	F	F F T T
F	F	T	F T F F
F	F	F	F T T T

- Next, rows for innermost connectives (\neg)
 - $\neg B$: flips value of B
 - $\neg C$: flips value of C
- Now, the row for the next innermost connective (\wedge)
 - \wedge takes lowest value

- Finally, the row for the main connective (\vee):
 - \vee takes highest value

You Try It

In Class Exercise

Break into **groups of 4-6** and construct a truth table for:

$$(4) (B \vee \neg C) \wedge \neg A$$

Boole

An Introduction

- *LPL* contains a program called **Boole**, which is for constructing truth tables
- Now that you've done a one by hand, you can appreciate how nice of a tool this is!
- Let's run through the basics of Boole by using it to construct a table for (3)

Truth Tables

Summary

- Okay, we've learned how to draw these pretty tables, but how do they help us do logic?
 - 1 Truth tables are a powerful way of exploring **logical possibility**
 - 2 This concept underlies several important concepts of logic:
 - Logical consequence
 - Logical equivalence
 - Logical truth
- We will now look at these concepts with the help of truth tables

Logical Truth

The Basics

Logical Truth

P is a **logical truth** if and only if it is **logically necessary**. That is, it is **not possible** for the laws of logic to hold while P is false

- Logical truths are those sentences which are guaranteed by logic alone to be true
- Logical possibility is different from physical & other kinds of possibility
 - $\text{Cube}(a) \vee \neg\text{Cube}(a)$ vs. $\text{Cube}(a) \vee \text{Tet}(a) \vee \text{Dodec}(a)$
 - Traveling the speed of light vs. being a round square
- This sounds vague, can we do any better?
 - Yes, if we use **truth tables**

Tautologies

Tautology

P is a **tautology** if and only if the truth table for P has only T's in the column under P's main connective

Truth Table for (1)

Cube(a)	Cube(a) \vee \neg Cube(a)
T	T
F	T

- So, (1) is a **tautology**
- Intuitively, is (1) a **logical truth**?
 - Yes!
- So, it looks like the idea of a tautology is a way of making the idea of a logical truth a bit more precise
- This is because truth tables are a precise way of thinking about **logical possibility**

Tautologies

Some Examples

- $\text{Cube}(c) \vee \neg\text{Cube}(c)$ is a tautology
- $\neg(\text{Tet}(a) \wedge \neg\text{Tet}(a))$ is a tautology
- $\text{Tet}(a) \vee \text{Cube}(a)$ is **not** a tautology

Tautologies vs. Logical Truths

- Recall that logical truths are sentences which are guaranteed to be true by the laws of logic alone
- All tautologies are logical truths
- But are all logical truths tautologies?
 - In other words, can we just replace the idea of a logical truth with that of a tautology?
- No! Seeing this actually takes a little creativity

A Curious Logical Truth

Which isn't a Tautology

- Surely it is a logical truth that Jay is Jay and that Kay is Kay:

$$(5) \quad j = j \wedge k = k$$

- But consider the truth table for (5):

Truth Table for (5)

$j = j$	$k = k$	$j = j \wedge k = k$
T	T	T
T	F	F
F	T	F
F	F	F

- First, reference columns
- Second, main connective
- But wait, there are F's in that column!
- When you build reference columns, you just list all the combinations

- But, some combinations don't make sense!

Tautologies & Logical Truths

Summary

Remember

- 1 P is a **tautology** if and only if every row of the truth table assigns T to P
- 2 If P is a tautology, then P is a logical truth
- 3 But some logical truths are not tautologies
- 4 P is **TT-possible** if and only if at least one row of its truth table assigns T to P

Let's do exercise 4.5

Summary

What we did today

- We learned how to construct truth tables, by hand & with Boole
- We then applied truth tables to the problem of precisely defining:
 - 1 Logical truth
- We came up with a similar concept:
 - 1 Tautology
- We saw that all Tautologies are logical truths, but some logical truths are **not** tautologies