

MODALITY AND MODAL LOGIC

1.30.12

WILLIAM STARR

Phil 6710 / Ling 6634 – Spring 2012

1 Introduction

- Big picture:
 - Explain why some claims follows from others
 - Use a formal language with a semantics:
 - ▶ Represent sentence used to make claim in a formal language
 - ▶ Give the formal language a semantics
 - ▶ Define semantic consequence (\models) relation for formal language
 - ▶ See whether it captures which claims follow from which
- Propositional logic:
 - **Syntax:**
 - ▶ Atomic formulas: P, Q, R, \dots , Connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
 - **Semantics:**
 - ▶ *Valuation:* v maps every atomic to a truth-value, 1 or 0
 - ▶ *Truth (wrt v) Definition:*
 - (1) $\llbracket A \rrbracket_v = 1 \iff v(A) = 1$
 - (2) $\llbracket \neg\phi \rrbracket_v = 1 \iff \llbracket \phi \rrbracket_v = 0$
 - (3) $\llbracket \phi \rightarrow \psi \rrbracket_v = 1 \iff \llbracket \phi \rrbracket_v = 0$ or $\llbracket \psi \rrbracket_v = 1$ \vdots
 - **Consequence:**
 - ▶ $\phi_1, \dots, \phi_n \models \psi \iff$ for every v : $\llbracket \psi \rrbracket_v = 1$ if $\llbracket \phi_1 \rrbracket_v = \dots = \llbracket \phi_n \rrbracket_v = 1$
- The important points:
 - Truth-value of complex formulas are determined by the truth-values of their parts
 - ▶ E.g. $\neg\phi$'s truth-value is determined by ϕ 's
 - Truth is relative to an assignment:
 - ▶ Only truth wrt a valuation v

Email: will.starr@cornell.edu.
URL: <http://williamstarr.net>.

- ▶ A valuation says how the 'atomic facts' come out
- ▶ It suffices to determine the truth value of every formula
- Consequence is defined in terms of truth

1.1 Modal Operators

- Consider *could*:
 - (1) Will could eat a bottle of glue (True)
- Is truth value of (1), determined by truth value of:
 - *Will ate a bottle of glue* (False)
- Can't be:
 - *Will ate the Earth* (False)
 - *Would could eat the Earth* (False)
- **The Point:**
 - Truth of modal sentences are not determined by the truth of their parts
 - It isn't possible to capture truth-conditions of modal claims in propositional logic!

1.2 Conditionals

- Indicative vs. Subjunctive:
 - Indicative:
 - (2) If Oswald didn't shoot Kennedy, someone else did
 - Subjunctive:
 - (3) If Oswald hadn't shot Kennedy, someone else would have
- Counterfactual:
 - Subjunctive conditional w/false antecedent
 - (4) If I were 7 feet tall, I would be taller than the Eiffel Tower
- Many problems for propositional logic analysis!
- **Problem 1** (Counterfactuals):
 - (4) is clearly false, but the corresponding material conditional is true!
- **Problem 2** (Logic):
 - The material conditional validates the following
 - Material Negation** $\neg(P \rightarrow Q) \models P$
 - Antecedent Negation** $\neg P \models P \rightarrow Q$

Contraposition $P \rightarrow Q \models \neg Q \rightarrow \neg P$

- But the first two are absurd and the last seems to have counterexamples:
 - (5) It is not true that if God exists, he is made of spaghetti. So God exists
 - (6) Bob didn't dance, so if Bob danced, he's a turnip.
 - (7) If it rains, it won't pour. So, if it pours it won't rain. (Adams 1975:15)
- Other problems pointed out by Lewis (1914:243-4)

2 Modal Logic

- **Goal:** a better analysis of conditionals and modal operators
- Propositional modal logic:
 - **Syntax:**
 - ▶ Atomic formulas: P, Q, R, \dots , Connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Box, \Diamond$
 - ▷ Strict Conditional: $\Box(\phi \rightarrow \psi)$
 - **Kripke Semantics:**
 - ▶ *Model:* $\mathcal{M} = \langle \langle W, R \rangle, v \rangle$
 - ▷ *Frame:* $\langle W, R \rangle$
 - ▶ *Set of Possible Worlds:* W , different complete ways the world could be
 - ▶ *Accessibility:* R , a set of pairs of worlds
 - ▷ R catalogs which worlds are possible with respect to each-other
 - ▷ $R(w, w')$ means that w' is possible wrt to w
 - ▶ *Valuation:* v maps every atomic and world to a truth-value at that world, 1 or 0
 - ▶ *Truth (wrt v, w, R) Definition:*
 - (1) $\llbracket A \rrbracket_{v,w}^R = 1 \iff v(A, w) = 1$
 - (2) $\llbracket \neg\phi \rrbracket_{v,w}^R = 1 \iff \llbracket \phi \rrbracket_{v,w}^R = 0$
 - (3) $\llbracket \phi \rightarrow \psi \rrbracket_{v,w}^R = 1 \iff \llbracket \phi \rrbracket_{v,w}^R = 0 \text{ or } \llbracket \psi \rrbracket_{v,w}^R = 1$
 - (4) $\llbracket \Box\phi \rrbracket_{v,w}^R = 1 \iff \text{for every } w' R(w, w') : \llbracket \phi \rrbracket_{v,w'}^R = 1$
 - (5) $\llbracket \Diamond\phi \rrbracket_{v,w}^R = 1 \iff \text{for some } w' R(w, w') : \llbracket \phi \rrbracket_{v,w'}^R = 1$
 - ⋮
 - **Consequence:**
 - ▶ $\phi_1, \dots, \phi_n \models \psi \iff \text{For every } \mathcal{M}, w: \llbracket \psi \rrbracket_{v,w}^R = 1 \text{ if } \llbracket \phi_1 \rrbracket_{v,w}^R = \dots = \llbracket \phi_n \rrbracket_{v,w}^R = 1$
 - ▷ $\mathcal{M} = \langle \langle R, W \rangle, v \rangle$ and $w \in W$

2.1 Motivating Kripke Semantics

- Why do we need R and W ?
- Let's try to do without them!
 - Aren't valuations possible worlds?
 - The way how every atomic fact is settled
 - Without W and R :
 - $\llbracket \Box\phi \rrbracket_v = 1 \iff \text{for every } v' : \llbracket \phi \rrbracket_{v'}^R = 1$ (Carnap 1956: §41)
 - This only captures logical necessity:
 - ▶ Necessities which are true at every world if they are true at any world!
 - ▶ But what about nomological necessity?
 - ▷ What is necessary according to the physical laws of our world may not be true of other worlds
 - ▶ If ϕ is logically necessary, it is true!
 - ▶ What about deontic necessity?
 - ▷ There are many things which are morally required, but don't come true
 - ▷ Yet, this semantics says that $\Box\phi \rightarrow \phi$!
 - R allows us to restrict the space of possibilities considered for the truth of $\Box\phi$
 - ▶ It thereby allows the formulation of necessity claims weaker than logical necessity
 - Without W (using valuations instead):
 - $\llbracket \Box\phi \rrbracket_v^R = 1 \iff \text{for every } v' R(v, v') : \llbracket \phi \rrbracket_{v'}^R = 1$
 - There's a problem about having enough worlds
 - Suppose there's only one possible object a
 - ▶ It can either be an apple $v_1(A) = 1$ or not $v_0(A) = 0$
 - Suppose you know A in v_1 : $\llbracket \Box A \rrbracket_{v_1}^R = 1$
 - But you don't know that you know, so: $\llbracket \Diamond\neg\Box A \rrbracket_{v_1}^R = 1$
 - ▶ Picture the accessibility relation!
 - ▶ v_1 can't see a $\neg A$ -world, but it must see a world w which can see a $\neg A$ -world
 - ▷ w must be an A world, but v_1 is the only A -world and it can't see a $\neg A$ -world
 - We need a third world to make $\Box A \wedge \Diamond\neg\Box A$ true!

2.2 Axioms and Systems

- Different constraints on R yield different logics for \Box and \Diamond
- Here's a roadmap of interesting constraints and logics

Relation Types	Definition
Reflexive	$\forall w : R(w, w)$
Symmetric	$\forall w, w' : R(w', w) \text{ if } R(w, w')$
Serial	$\forall w, \exists w' : R(w, w')$
Transitive	$\forall w, w', w'' : R(w, w') \ \& \ R(w', w'')$

Name	Axiom
K	$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
T	$\Box\phi \rightarrow \phi$
B	$\phi \rightarrow \Box\Diamond\phi$
D	$\Box\phi \rightarrow \Diamond\phi$
4	$\Box\phi \rightarrow \Box\Box\phi$
E	$\Diamond\phi \rightarrow \Box\Diamond\phi$

System	Axioms	Accessibility
K	K	Any $W \times W$
T	KT	Reflexive
B	KTB	Reflexive & Symmetric
D	KD	Serial
S4	KT4	Reflexive & Transitive
S5	KTE	Symmetric & Transitive

2.3 Strict Conditionals

- Problems for material conditional:

Material Negation $\neg(P \rightarrow Q) \vDash P$

Antecedent Negation $\neg P \vDash P \rightarrow Q$

Contraposition $P \rightarrow Q \vDash \neg Q \rightarrow \neg P$

- Facts about strict conditional:

Material Negation Invalid $\neg\Box(P \rightarrow Q) \not\vDash P$

Antecedent Negation Invalid $\neg P \not\vDash \Box(P \rightarrow Q)$

Contraposition $\Box(P \rightarrow Q) \vDash \Box(\neg Q \rightarrow \neg P)$

Necessary Antecedent Negation $\Box\neg P \vDash \Box(P \rightarrow Q)$

- Lewis and Stalnaker will raise more problems in Week 4.

2.4 Philosophical Issues

- *Actualism*: there's only one possible world, the actual one!
 - What about it from bit physics (Wheeler)? Too Wacky?

- *Reduction / Truth-making*: if modal truths were reducible to non-modal ones, you would expect possible worlds and valuations to be the same thing
 - How can our world make modal claims true, if modal logic's truth conditions obtain?
 - ▶ It requires our world to bear relations to other worlds
 - ▶ And that doesn't seem like something that just our world can do!
- *Instrumentalism*:
 - W doesn't really contain possible worlds, remember, we don't make any assumption about it's elements
 - ▶ Just that they are there!
- *Communication*:
 - On the modal logic analysis, modal claims only have truth-values relative to v, w, R
 - But where do these come from?
 - Context, you might say, but what exactly are contexts such that they would provide *these* things?
 - Can this capture the idea that different kinds of uses of a modal word exist in the same language?

3 Modal Logic and Natural Language

Simple Modal Logic Hypothesis The meaning of every modal expression in natural language can be expressed in terms of only two properties

- Whether it is a necessity or possibility modal, and
- Its accessibility relation, R .

3.1 Kinds of Modality

- Some basic kinds:

Nomological (Natural Law) The feather and bowling ball must fall at the same speed.

Metaphysical I could have been a doctor.

Epistemic (Knowledge/Belief) It must be raining.

Deontic (Moral Duty) John must be punished.

Ability Will can eat a bottle of glue.

Bouletic (Desire) I must have some coffee.

- Different accessibility relations for each?

-
- Treatment *must* and *should* in Portner:
 - Doesn't this make the difference between *must* and *should* the same as that between different flavors of modality?

3.2 Limitations of Modal Logic Analysis

- Within the kinds of modality, we find specific modifiers:
 - In view of the traffic laws, you must not speed.
- Mixtures of different accessibility relations in a conjunction:
 - I ought to donate to charity, and I might do so
- How do these parameters connect to what we ordinarily think of the context of an utterance?

References

- ADAMS, EW (1975). *The Logic of Conditionals*. Dordrecht: D. Reidel.
- CARNAP, R (1956). *Meaning and Necessity*. 2 edn. Chicago: Chicago University Press. (First edition published in 1947.)
- LEWIS, CI (1914). 'The Calculus of Strict Implication.' *Mind*, **23(90)**: 240–247. URL <http://www.jstor.org/stable/2248841>.